Agreement testing theorems on layered set systems

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***Abstract*—We introduce a framework of layered subsets, and give a sufficient condition for when a set system supports an agreement test. Agreement testing is a certain type of property testing that generalizes PCP tests such as the plane vs. plane test. Previous work has shown that high dimensional expansion is useful for agreement tests. We extend these results to more general families of subsets, beyond simplicial complexes. These include**

- **Agreement tests for set systems whose sets are faces of high dimensional expanders. Our new tests apply to all dimensions of complexes both in case of two-sided expansion and in the case of one-sided partite expansion. This improves and extends an earlier work of Dinur and Kaufman (FOCS 2017) and applies to matroids, and potentially many additional complexes.**
- **Agreement tests for set systems whose sets are neighborhoods of vertices in a high dimensional expander. This family resembles the expander neighborhood family used in the gap-amplification proof of the PCP theorem. This set system is quite natural yet does not sit in a simplicial complex, and demonstrates some versatility in our proof technique.**
- **Agreement tests on families of subspaces (also known as the Grassmann poset). This extends the classical low degree agreement tests beyond the setting of low degree polynomials.**

Our analysis relies on a new random walk on simplicial complexes which we call the "complement random walk" and which may be of independent interest. This random walk generalizes the non-lazy random walk on a graph to higher dimensions, and has significantly better expansion than previously-studied random walks on simplicial complexes.

***Index Terms*—agreement, Direct Product Test, PCP**

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I. Introduction

Agreement testing is a certain type of property testing. The first agreement testing theorems are the line versus line or plane versus plane low degree agreement tests [1], [2], [3] that play an important part in various PCP constructions. We discuss the history and evolution of these tests further below.

Abstractly, an agreement test is the following. Let V be a ground set and let S be a family of subsets of V. The object being tested is an ensemble of local functions $\{f_s \in \Sigma^s \mid s \in S\}$ with one function per set $s \in S$. The domain of f_s is s itself. A *perfect ensemble* is an ensemble that comes from a global function $g : V \to \Sigma$ whose domain is the entire vertex set. In a perfect ensemble the local function at s is the restriction of g to the set s, that is, $f_s = g\restriction_s$ for all $s \in S$.

We let $\mathcal{G}$ be the set of all perfect ensembles. An agreement test is a property tester for $\mathcal{G}$. It is specified by a distribution over pairs[1] of intersecting subsets, $s_1, s_2 \in S$, and the test accepts if the respective local functions agree on the intersection: $f_{s_1}\restriction_t = f_{s_2}\restriction_t$ where $t = s_1 \cap s_2$. A perfect ensemble is clearly accepted with probability 1. The test is *c-sound* if

$$\mathrm{dist}(f, \mathcal{G}) \leqslant c \cdot \mathop{\mathbb{P}}_{s_1, s_2}[f_{s_1}\restriction_t = f_{s_2}\restriction_t]. \tag{I.1}$$

Here the distance $\mathrm{dist}(f, \mathcal{G})$ is the minimal fraction of sets $s \in S$ that we need to change in f in order to get a function in $\mathcal{G}$.

It is well known (see Example II.3) that in some cases exact soundness is impossible and we must allow a slightly weaker notion, called γ-approximate soundness. The γ-approximate distance between two ensembles f and g, denoted $\mathrm{dist}_\gamma(f, g)$, is the fraction of sets s in which $\mathrm{dist}(f_s, g_s) > \gamma$. An agreement test is *$\gamma$-approximately c-sound* if

$$\mathrm{dist}_\gamma(f, \mathcal{G}) \leqslant c \cdot \mathop{\mathbb{P}}_{s_1, s_2}[f_{s_1}\restriction_t = f_{s_2}\restriction_t]. \tag{I.2}$$

This means that if the test succeeds with probability $1 - \varepsilon$ there must be a global function $g : V \to \Sigma$ such that for all but $c \cdot \varepsilon$ of the sets s, $\mathrm{dist}(f_s, g\restriction_s) \leqslant \gamma$.

a) Why study agreement tests.: The original motivation for agreement tests comes from PCP proof composition: a key step in this construction is to combine many small proofs into one global proof, but without knowing whether the small proofs are consistent with each other. The agreement test ensures that they can be combined together coherently. Indeed, agreement tests are the basis of the "inner verifier" constructed in recent works on $2:2$ games [5], [6], [7], [8].

Recent work [9] used agreement tests in a different context, for proving structure theorems for Boolean functions. The idea is to prove structure for small restrictions of the function, often an easier task, and then apply an agreement testing theorem to combine these structures together.

Agreement tests are a natural family of tests that seems interesting in its own right. This work makes a step towards developing a theory that explains which set systems have agreement tests.

The STAV layered set system

We describe a three layered set system which we call a STAV.

Looking closely at agreement tests, we can always model them with three layers: the vertices (V), the sets (S) and the possible intersections between sets (T). The STAV has an additional so-called "Amplification" layer (A) that captures an amplification property that occurs in many interesting settings: given that we know that two local functions agree on part of the intersection, the probability that they will agree on the whole intersection rises significantly.

We give an informal description of STAV, for the detailed formal definition please see Section II-B. A STAV is a tuple (S, T, A, V) together with the following three distributions

- The STAV distribution - a distribution over (s, t, a, v), $s \supset t \supset a \cup v$.
- The STS distribution - a distribution over s_1, t, s_2 that gives the agreement testing distribution and in addition a subset $t \subseteq s_1 \cap s_2$.

[1] In some cases the test can query more than two subsets, as in the so-called Z-test of [4], but in this paper we restrict attention only to two query tests.

– The VASA distribution - a distribution over v, a, s, a' whose role will be made clear in the analysis.

A STAV is called γ-good if these distributions (and some local views of them) satisfy certain spectral conditions.

b) The surprise *parameter.:* Based on the STAV structure, it is natural to define a parameter which we call the surprise. This parameter depends both on the ensemble $f = \{f_s\}$ and on the STAV, and in some cases, it can be bounded independently of f (this is the case for simplicial complexes). The surprise parameter is a measure of how much amplification the A layer gives us. It is the probability that two intersecting sets agree on a given that they disagree on t (See Definition II.17). This parameter gives a unified way to address different agreement scenarios.

Main Results

This paper is short version, therefore some of the results and proofs are ommited. For the full version of this paper, see [10].

Our main technical theorem (Theorem II.26) says that every set system that supports a γ-good STAV must support a sound agreement test. This reduces the task of proving an agreement test to the much simpler task of uncovering a STAV underneath the set system.

We list here a few applications of this theorem, starting with agreement tests for high dimensional expanders. Introducing high dimensional expanders is beyond the current scope and we refer the reader to Section C for more introductory definitions.

Theorem I.1 (Agreement for two-sided HDX)**.** *There exists a constant $c > 0$ such that for every d-dimensional simplicial complex X the following holds. If X is a $\frac{1}{d^3}$-two-sided d-dimensional HDX, then $X(d)$ supports a c-sound agreement test.*

In the full paper we describe some corollaries of this theorem for matroids.

The only known constructions of *sparse* two-sided HDXs are by truncating one-sided HDXs, see the Ramanujan complexes of [11] as well as the construction of HDXs due to [12]. It is natural to study agreement tests for the (non-truncated) one-sided HDX itself. The following theorem gives such a result in the special case that the complex is also $d+1$-partite. Many Ramanujan complexes are naturally $d+1$-partite, as are the complexes constructed in [12].

Theorem I.2 (Agreement for partite one-sided HDX)**.** *There exists a constant $c > 0$ such that the following holds. Suppose X is a $(d+1)$-Partite complex that is a $\frac{1}{d^3}$-one sided HDX. Then $X(d)$ supports a c-sound agreement test.*

Our next agreement theorem is for a family of subsets that is derived from a high dimensional expander, although itself it does not sit inside a simplicial complex. The subsets in this family are balls, or neighborhoods, of a vertex or a higher dimensional face in a simplicial complex that is a HDX. This construction resembles the set system underlying the gap-amplification based proof of the PCP theorem [13], in which an agreement theorem underlies the argument somewhat implicitly.

Theorem I.3 (Agreement on neighborhoods)**.** *There exists a constant $c > 0$ such that the following holds. Let X be a $\frac{1}{d^3}$-two-sided high dimensional expander. For each vertex $z \in X(0)$ let B_z be the set of neighbors of z, and let $S = \{B_z \mid z \in X(0)\}$. Then S supports a $\frac{1}{d}$-approximately c-sound agreement test.*

Finally, our last agreement theorem is for a family of subspaces of a vector space, also called the Grassmann. Such families were studied in PCP constructions for special ensembles whose local functions belong to some code. Such ensembles are guaranteed to have the following property. For all s_1, t, s_2, if $f_{s_1}\restriction_t \neq f_{s_2}\restriction_t$ then $\mathrm{dist}(f_{s_1}\restriction_t, f_{s_2}\restriction_t) \geqslant \delta$. We call such ensembles δ-ensembles and prove,

Theorem I.4 (Agreement on subspaces - informal)**.** *There exists a constant $c > 0$ such that the following holds. Let $\mathbb{F}^n$ be a vector space and let S have a set for every affine subspace of dimension d. Then S supports a $1/q^{\Omega(d)}$-approximately c-sound* *agreement test for δ-ensembles.*

For the benefit of the reader we added in Section C a list of theorems proven in this work.

Overview of the proof of our main theorem (Theorem II.26)

Our main agreement theorem on STAV structures has two parts, as in many previous works. The first part of the proof uses the amplification given by the surprise parameter to construct a family of functions for each $a \in A$, that is $\mathbf{g} = \{g_a : \text{ reach}_a \to \Sigma \mid a \in A\}$. The reach of a is the set of all vertices v, so that $\{v\} \mathbin{\dot\cup} a \subset s$ for some $s \in S$. The value $g_a(v)$ is defined by popularity of $f_s(v)$ for all $s \supset a$. This part is standard and occurs in many agreement test analyses.

The second part of the proof is our main new technical contribution. In this step one constructs a global $G : V \to \Sigma$ from the pieces g_a. This is done by showing sufficient agreement between the different g_a's. We consider a graph connecting a pair a, a' when they sit together inside some s. In earlier works this graph is dense and has very low diameter (2 typically). This can only happen when the functions g_a are defined on a pretty large part of the vertex set (as in [14], [15], [9], [3]) unlike our context where each reach_a is quite tiny (its size can be a constant, far smaller than $|V|$). When the diameter is small and reach_a is huge it is easy to stitch the different g_a's together, even when the agreement between the g_a's is rather crude, by taking a very short random walk from a to a' to a''.

In contrast, in our case the diameter is logarithmic and we cannot afford a random walk because the error would build up badly. Instead, we construct the global function $G : V \to \Sigma$ by

$$G(v) = \text{pop}\{g_a(v) \mid a \in \text{ reach}_v\},$$

i.e. the most popular opinion of the g_a's on v. We show that it has the desired properties. This argument relies on the fact that the VASA random walk (in particular, moving from a to s to a') is a very strong expander. That such VASA distributions are available is proven through a new type of random walk which we call the complement random walk, and is discussed separately below.

The only previous work that analyzed an agreement test on a sparse set system (where this "large diameter" problem appears) was in [16]. Their solution circumvented this problem by reducing to the dense case in a certain way. That reduction is ad-hoc and required an additional external layer of sets above S, which limited the generality of the theorem. Whereas the current proof is more direct and works without this technical caveat.

The complement random walk in high dimensional expanders: Several previous works [17], [16], [18] analyzed random walks on high dimensional expanders[2]. In this work we study a new type of random walk which we call the complement random walk.

Interestingly, independent recent work of Alev, Jeronimo, and Tulsiani [19], studies the same walk, where it is called "swap walk". The authors use this walk for analyzing an algorithm that solves constraint satisfaction problems (CSPs) on high dimensional expanders.

The complement walk goes from i-face to i-face via a shared j-face, just like the upper and lower random walks previously studied. However it has significantly better expansion, and is hence much more useful for us. We construct with it γ-good STAVs in many of our applications. The problem with many of the previously studied random walks is that they have an inherent "laziness" built in: starting from an i face and walking down to a j face, and then back up to another i face, the $j + 1$ common vertices are limiting the expansion of this walk (the family of all sets containing a fixed vertex will have not-so-good expansion). In contrast, the complement walk starts with an i-face a moves up to a j-face $b \supset a$ and then moves down to another i-face $a' \subset b$ conditioned on a, a' being *disjoint* (of course we need $j \geqslant 2i + 1$, note that any choice of such j would give the exact same random walk). It turns out (see Theorem A.21) that this

[2]In this section we assume familiarity with high dimensional link expansion, see Section C for formal definitions.

walk has great expansion. This can be seen by examining for example the case of $i = 0$ and noting that this is just the non-lazy random walk on a graph.

We prove the properties of this (and other) walks in the full paper. The proof goes through *Garland's method*. This method, proves global properties of the simplicial complexes by properties on the links. This method, originally developed by Garland in [20], is used in many works such as [21], [16], [22].

We believe these random walks are interesting on their own account. These walks generalize the non-lazy adjacency operator in a graph, and the bipartite adjacency operator in a bipartite graph to high dimensions. As a bonus we show an immediate application for these walks: a new high dimensional expander mixing lemma for sets in all dimensions , extending the work of [23], [24].

More background and context

As mentioned earlier the first agreement testing theorems are the line versus line or plane versus plane low degree agreement tests [1], [2], [3] that play an important part in various PCP constructions. Combinatorial analogs of these theorems were subsequently dubbed "direct product tests" and studied in a sequence of works [25], [26], [27], [4], [14], [28]. For a long while there were only two prototypical set systems for which agreement tests were known:

- All k-dimensional subspaces of some vector space
- All k-element subsets of an underlying ground set

Each of these has several variants (varying the field size and ambient dimension, deciding whether the sets are ordered or not, etc.).

The study of agreement tests initially came as a part of a PCP construction, as in the case of the low degree agreement tests and later in works leading towards combinatorial proofs for the PCP theorem, as started in [25] and continued in [26], [13].

Further works relied on agreement tests for hardness amplification: [4] showed hardness for label cover (called a two-query PCP) based on their direct product agreement test. A recent line of work [5], [6], [7], [8] concerning unique and $2:2$ games used agreement tests on the Grassmann as an inner verifier (see in particular [6]).

In hope of getting more efficient PCPs and LTCs it seemed that understanding the power of agreement tests in a more general setting would give us a better handle on domains in which locally testable codes and PCP constructions can reside. However, despite some attempts, no derandomization techniques managed to find further (and hopefully sparser) constructions.

A couple of years ago [16] discovered a new and very sparse set system that supports an agreement test. This new system is based on group theoretic (and number theoretic) constructions of so-called high dimensional expanders. The number of sets in this set system is linear in the size of the ground set, a feature that seems key towards new and more efficient locally testable codes and PCPs.

This suggested that there is possibly a much richer collection of set systems that support agreement tests, and brought to the fore once more the question of understanding which set systems support agreement tests.

II. Agreement Tests for STAV Structures

A. Agreement tests and agreement expansion

We begin with the definition of an agreement expander, similar to that of [16]. Let S be a family of subsets of a ground set V. An ensemble of local functions is a collection $\{f_s : s \to \Sigma \mid \ s \in S\}$ consisting, for each subset $s \in S$, of a function whose domain is s. A perfect ensemble is one that comes from a global function $g : V \to \Sigma$, namely $f_s = g\restriction_s$ for all $s \in S$. We denote the set of all perfect ensembles by

$$\mathcal{G}(V;\Sigma) = \{\{g\restriction_s\}_{s\in S} \mid \ g : V \to \Sigma\}\,.$$

An agreement test is given by a distribution $\mathcal{D}$ over pairs of intersecting subsets,

- Input: An ensemble of local functions $\{f_s : s \to \Sigma \mid \ s \in S\}$

– Test: Choose a random edge $\{s_1, s_2\}$ according to the distribution $\mathcal{D}$, let $t = s_1 \cap s_2$ and accept iff $f_{s_1}\restriction_t = f_{s_2}\restriction_t$.

We denote by $\mathrm{rej}_{\mathcal{D}}(f)$ the probability that the agreement test rejects a given ensemble $f = \{f_s\}$. A perfect ensemble is clearly accepted with probability 1. We say that the test is sound if it is a sound test for the property $\mathcal{G}(V; \Sigma)$ in the standard property testing sense, namely,

Definition II.1 (Sound agreement test)**.** An agreement test is c-sound if every ensemble $f = \{f_s\}$ satisfies

$$\mathrm{dist}(f, \mathcal{G}) \leqslant c \cdot \mathrm{rej}_{\mathcal{D}}(f)\,.$$

Finally we can define an agreement expander,

Definition II.2 (c-agreement expander)**.** An agreement expander is a family S of subsets of a ground set V that supports a c-sound agreement test.

The reason for the term "agreement expander" is the similarity to a Rayleigh quotient given by

$$\frac{1}{c} = \inf_{f \notin \mathcal{G}} \frac{\mathrm{rej}_{\mathcal{D}}(f)}{\mathrm{dist}(f, \mathcal{G})}\,,$$

where the numerator counts the number of rejecting edges and the denominator measures the distance from the property. See [29] for a more detailed analogy between expansion and property testing.

Approximate versus exact agreement

For some agreement tests one cannot expect a conclusion as strong as in Definition II.2. For example, suppose that the testing distribution $\mathcal{D}$ selects pairs s_1, s_2 that typically intersect on an $\eta \ll 1$ fraction of s_1 (and of s_2). In such a case consider the following ensemble,

Example II.3. Construct an ensemble $f = \{f_s\}$ at random as follows. For all s set $f_s = 0\restriction_s$ and then for each s with probability α do: change one bit of f_s at random.

This ensemble passes the test with probability at least $1 - 2\alpha\eta$ while being roughly α-far from $\mathcal{G}$. Setting $\alpha = 1$ rules out any kind of conclusion as in Definition II.2. However, not all is lost, and a meaningful theorem can still be proven if we move to a softer notion of *approximate* agreement. Let us denote by $\mathrm{dist}_\gamma(f, f')$ the fraction of sets s on which f_s, f'_s differ on more than γ fraction of s. Namely,

$$\mathrm{dist}_\gamma(f, f') = \mathop{\mathbb{P}}_s[\mathrm{dist}(f_s, f'_s) > \gamma].$$

Definition II.4 (γ-approximate soundness)**.** An agreement test is γ-approximately c-sound if every ensemble $f = \{f_s\}$ satisfies

$$\mathrm{dist}_\gamma(f, \mathcal{G}) \leqslant c \cdot \mathrm{rej}_{\mathcal{D}}(f)\,.$$

When $\gamma < 1/|s|$ we recover the previous notion of soundness which we now call *exact* soundness. So a test is c-sound or exactly c-sound if it is γ-approximately c-sound for some $\gamma < 1/|s|$.

B. STAV structures

A STAV structure introduces two additional layers of subsets of V: layer T and layer A. These come in addition to the top layer S that we already have in the definition of an agreement expander. The layer T represents the intersections of pairs of subsets $s_1, s_2 \in S$, and is implicit in the definition of the agreement test distribution. The layer A is new and sits below T. It provides a certain amplification needed for the analysis.

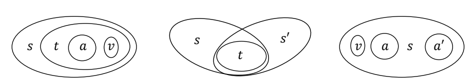

Figure 1. The STAV, STS, and VASA distributions

Definition II.5 (STAV structure)**.** A STAV structure is a tuple $X = (S, T, A, V;\, D_{\mathrm{STAV}})$ consisting of a ground set V and three layers of subsets $A, T, S \subset \mathbb{P}(V)$, together with a stochastic process D_{STAV} that samples (s, t, a, v) as follows.

– Choose s
– Choose t conditioned on s
– Choose a, v conditioned on t (but not dependent on s)

The distributions in which the above are chosen are not restricted except for assuming that the marginal of this process is uniform over v and that the probability to choose a vertex or a set is never zero. The STAV comes with two distributions,

- *STS distribution:* A distribution over triples (s_1, t, s_2) that is symmetric with respect to s_1, s_2 and satisfies that the marginal of (s_1, t) (and therefore (s_2, t)) is identical to the marginal of D_{STAV}.
- *VASA distribution:* A distribution D_{VASA} over tuples (v, a_1, s, a_2) that is symmetric with respect to a_1, a_2 and satisfies that the marginal of (v, a_1, s) (and therefore (v, a_2, s)) is identical to the marginal of D_{STAV}.

Notation: Throughout this paper we use the letters s, t, a, v to denote elements in S, T, A and V respectively without specifically mentioning this. So for example fixing a_0, $\{s \supset a_0\}$ stands for all elements of S that contain $a_0 \in A$. Unless specified otherwise, all random choices are with respect to the distributions D_{STAV} or the STS or VASA distributions.

Before we continue to define what a "good" STAV is, let us mention a couple of examples that might be useful to keep in mind.

Example II.6 (The direct product test STAV)**.** Fix k and let $\ell = k/3$. We construct the following family of STAVs for all $n \gg k$, $n \to \infty$. Let $V = [n]$, let $S = \binom{[n]}{k}, T = \binom{[n]}{\ell}$ and $A = \binom{[n]}{\ell-1}$. The STAV distribution is choosing a k-element set uniformly, then an ℓ-element subset of it, and then splitting t randomly into a and v. A possible STS distribution is to choose a random t and then two independent $s_1, s_2 \supset t$. Another possibility is to choose $s_1, s_2 \supset t$ so that their intersection is exactly t. The VASA distribution is to choose s uniformly and in it a, a', v uniformly so that they are all disjoint.

An agreement test for this example appears in [14] under the name direct product test.

Example II.7 (HDX simplicial complexes, generalizing Example II.6)**.** Fix k and let $\ell = k/3$. We construct the following family of STAVs for infinitely many $n \gg k$. Suppose X is a high dimensional expander on n vertices. Let $V = X(0)$, let $S = X(k), T = X(\ell)$ and $A = X(\ell - 1)$. The STAV distribution is choosing a random s from the distribution of X, then a uniform $t \subset s$, and then splitting t randomly into a and v. A possible STS distribution is to choose a random t and then two independent $s_1, s_2 \supset t$. Another possibility is to choose $s_1, s_2 \supset t$ so that they must be disjoint. The VASA distribution is to choose s according to the X distribution and in it a, a', v uniformly so that they are all disjoint.

Agreement tests for this example were analyzed in [16] for certain complexes X and certain bounds on the dimension k.

Example II.8 (Subspaces STAV)**.** Fix $m > d > \ell$. We construct the following family of STAVs for all finite fields $\mathbb{F} = \mathbb{F}_q$, $q \to \infty$. Let $V = \mathbb{F}^m$, let S be all d-dimensional spaces of V, let T be all ℓ-dimensional spaces of V and let A be all $(\ell - 1)$-dimensional spaces of V. The STAV distribution is choosing s uniformly, $t \subset s$ uniformly, then $a \subset t$ uniformly, then v uniformly from $t \setminus a$. A possible STS distribution is to choose a random t and then two uniform $s_1, s_2 \supset t$. The VASA distribution is to choose s uniformly and in it a, a', v uniformly so that they are all disjoint.

This example generalizes the plane vs. plane low degree agreement test. An agreement test for it is proved in [3] for ensembles whose local functions are low degree functions, and in [4] for general ensembles (in both cases the focus was on a different parameter regime).

We now define several graphs that arise as local views of the STS and VASA distributions. The first of these is the bipartite graph obtained by the marginal of D_{STAV} on A and V,

Definition II.9 (The *AV*-Graph (reach graph))**.** The *AV*-graph, or reach graph, is a bipartite graph (V, A, E) where the probability of choosing an edge (v, a) is given by the marginal of D_{STAV} on $V \times A$, namely, $Pr[(v, a)] = \sum_{s,t} \mathbb{P}_{D_{\text{STAV}}}[(s, t, a, v)]$.

We denote $\text{reach}_a \subset V$ the set of neighbors of a in this graph, and by

$\text{reach}_v \subset A$ the set of neighbors of v in this graph.

Definition II.10 (The local reach graphs)**.** Let X be a STAV-structue, and fix $s \in S$. The *s-local* reach graph, or AV_s-graph, is a bipartite graph where:

$$L = \{a \mid \ a \subset s\}.$$

$$R = \{v \mid \ v \in s\}.$$

$$E = \{(a, v) \mid \ v \in \ \text{reach}_a\}.$$

The probability of choosing an edge (a, v) is the probability of choosing (a, v) in the STAV-distribution given that we chose s.

The STS graph and its local views: The STS distribution is conveniently viewed as a graph whose vertex set is S and whose edges are labeled by elements of T, with the weight of the edge from s_1 to s_2 labeled by t given by the probability of (s_1, t, s_2). The graph is undirected since the STS distribution is symmetric wrt s_1, s_2.

We consider "local views" of the STS graph - obtained by inducing it on a smaller set of vertices.

Definition II.11 (STS_a-Graph)**.** For a fixed a, an STS_a-Graph is has vertex set $\{s \mid \ s \supset a\}$ and the probability of choosing an edge $\{s_1, s_2\}_t$ is given by $2\,\mathbb{P}_{\text{STS}}\left[(s_1, t, s_2) \mid \ t \supset a\right]$.

Definition II.12 ($\text{STS}_{a,v}$-Graph)**.** For a fixed a, v, an $\text{STS}_{a,v}$-Graph is has vertex set $\{s \mid \ s \supset a \cup \{v\}\}$ and the probability of choosing an edge $\{s_1, s_2\}_t$ is given by $2\,\mathbb{P}_{\text{STS}}\left[(s_1, t, s_2) \mid \ t \supset a \cup v\right]$.

Local views of the VASA distribution: When fixing one of the four terms in (v, a, s, a'), we can define the following two graphs by the marginal:

Definition II.13 (${}_vASA$-Graph)**.** For a fixed v, an ${}_vASA$-Graph is the graph whose vertex set is reach_v, and labeled edges are $E = \{\{a_1, a_2\}_s \mid \ a_1, a_2 \in A, s \in S, v, a_1, a_2 \subset s\}$. The probability to choose an edge $\{a_1, a_2\}_s$ is given by $\mathbb{P}_{D_{\text{VASA}}}\left[(v', a_1, s, a_2) \mid \ v' = v\right]$.

Definition II.14 (Bipartite VAS_a-Graph)**.** For a fixed a, an VAS_a-Graph is the bipartite graph (L, R, E) where

$$L = \ \text{reach}_a,$$
$$R = \left\{(a', s) \mid \ \exists v \in L\ (v, a, s, a') \in \mathit{Supp}(D_{\text{VASA}})\right\},$$
$$E = \left\{(v, (a', s)) \mid \ (v, a, s, a') \in \mathit{Supp}(D_{\text{VASA}})\right\}.$$

The probability of choosing an edge $(v, (a', s))$ is given by $\mathbb{P}_{D_{\text{VASA}}}\left[(v, a_0, s, a') \mid \ a_0 = a\right]$.

Good STAV-Structures: Having defined all the relevant graphs, we come to the requirements for a good STAV:

Definition II.15 (A good STAV-Structure)**.** Let X be STAV structure and $\gamma < 1$ be some constant. We say X is a γ-good if assumptions (A1)-(A3) and one of (A4(r)) or (A4) below hold for X:

(A1) The reach graph is a $\sqrt[3]{\gamma}$-bipartite expander.

(A2) a) For all $a \in A$, the STS_a-Graph is a $\frac{1}{3}$-edge expander.
b) For all $a \in A$ and $v \in \ \text{reach}_a$, the $STS_{(a,v)}$-graph is an γ-two-sided spectral expander.

(A3) a) For all $v \in V$, the ${}_vASA$-graph is a either a γ-bipartite expander or a γ-two-sided spectral expander.
b) For all $a \in A$, the VAS_a-graph is a $\sqrt{\gamma}$-bipartite expander.

(A4(r)) For all $s \in S$, the AV_s-graph is a $r\gamma$-sampler graph. Here $r > 0$ is a parameter. A $r\gamma$-sampler graph is defined in Definition A.5.

(A4) For every pair a, s so that $a \subset s$, the size of reach_a inside s is relatively large, that is

$$\mathop{\mathbb{P}}_{v \sim D}\left[v \in \ \text{reach}_a \mid \ v \in s\right] \geqslant \frac{1}{2}.$$

Remark II.16. The constants $\frac{1}{2}, \frac{1}{3}$ are arbitrary. In addition, in the proof of the main theorem, we will use the fact that the graphs in Assumption **(A3)**, Assumption **(A2)**b are $\frac{1}{3}$-edge expanders. By the famous Cheeger's inequality, for a small enough γ, if the graphs above are γ-spectral expanders, then they are also $\frac{1}{3}$-edge-expanders.

C. The surprise parameter

Let $f = \{f_s\}_{s \in S}$ be an ensemble. In this section we discuss an additional parameter of f and the underlying STAV structure X that influences the agreement theorem. This is the so-called *surprise* parameter. This parameter measures how surprised we are when f_s and $f_{s'}$ agree on a given that we already know that they disagree on t, where $t \supset a$. If this probability is small, we get strong amplification. This idea played an important role in several previous works and it seems useful to consider this parameter explicitly.

Definition II.17 (Surprise of an ensemble)**.** Let X be a STAV structure. The surprise of a given ensemble $f = \{f_s\}$ with respect to X is

$$\xi(X, f) = \mathop{\mathbb{P}}_{s_1,s_2,t,a,v}[f_{s_1}\restriction_a = f_{s_2}\restriction_a \text{ and } f_{s_1}(v) \neq f_{s_2}(v) \\ \big| f_{s_1}\restriction_t \neq f_{s_2}\restriction_t]$$

where the probability is over choosing s_1, t, s_2 from the STS distribution and then choosing (a, v) conditioned on t. Note that both s_1, t, a, v and s_2, t, a, v are distributed as in D_{STAV}.

It is sometimes natural to restrict attention to a sub-family of ensembles which we call δ-ensembles.

Definition II.18 (δ ensemble)**.** An ensemble f is a δ-ensemble if for every labeled edge (s_1, t, s_2) in the STS graph,

$$f_{s_1}\restriction_t \neq f_{s_2}\restriction_t \implies \operatorname{dist}(f_{s_1}\restriction_t, f_{s_2}\restriction_t) > \delta$$

(where $\operatorname{dist}(\cdot,\cdot)$ stands for relative hamming distance).

Remark II.19*.* Note that every ensemble is a $\frac{1}{|t|}$ ensemble.

Remark II.20*.* Agreement theorems are often considered for special ensembles where each f_s belongs to an error correcting code, such as the Reed-Muller code in the case of low degree tests. Furthermore, in the low degree test examples, for all $t \subset s$, $f_s\restriction_t$ itself belongs to an error correcting code with some distance δ. Clearly, such ensembles are automatically δ-ensembles.

In some important cases the STAV structure itself implies a non-trivial surprise parameter for all possible ensembles. We are thus led to define the surprise of the STAV as the supremum over all possible ensembles,

Definition II.21 (Global surprise)**.** Let X be a STAV structure. The surprise of X is

$$\xi(X) = \sup_f \xi(f)\,.$$

While the agreement of f is a property of the ensemble f, the surprise is influenced by the STAV-structure itself. For this, the following graphs play a role:

Definition II.22 (T-Lower Graph)**.** Fix $t \in T$. The T-lower graph of t is a bipartite graph where

$$L = \{v \mid\ v \in t\}\,,\ R = \{a \mid\ a \subset t\}\,,\\ E = \{(a, v) \mid\ v \in a\}\,.$$

Notice that here, we require $v \in a$ and *not* $v \in \text{reach}_a$ as we required in the STAV-structure. The probability to choose an edge $(a, v) \in E$ is the probability of choosing a given that $a \subset t$ and then choosing v at random inside a.

A priori, the T-lower graphs need not be good expanders, as in the STAV-structures defined for one-sided high dimensional expanders. However, when they are, we can use their expansion properties to establish the "surprise". We can give the following easy bound on the surprise parameter,

Lemma II.23. *Let X be a STAV-structure so that for every $t \in T$, the T-lower graph is a η-bipartite expander. For any δ ensemble f, $\xi(X, f) \leqslant O(\frac{\eta^2}{\delta})$.*

Before proving the lemma let us give a couple of examples demonstrating its usefulness.

Example II.24 (HDX simplicial complexes, continued)**.** Consider the STAV from Example II.7. For any $t \in X(\ell)$, the T-lower graph of t is the graph where R is the vertices of t, and L are subsets of t of size $|t| - 1$, where the edges denote containment. The reader may calculate that this graph is a η-bipartite expander with $\eta = \frac{1}{\ell}$. Plugging in $\delta = 1/\ell$ we get $\xi(X) \leqslant \eta^2/\delta = 1/\ell$.

Example II.25 (The Grassmann Poset)**.** Let $\mathbb{F}$ be a finite field, let X is a STAV-structure where $V = \mathbb{F}^n$, T is the set of ℓ-dimensional linear subspaces of $\mathbb{F}^n$, A is the set of $(\ell-1)$-dimensional spaces. For any $t \in X(\ell)$, the T-lower graph of t is the graph where R are the 1-dimensional subspaces of t, and L are the ℓ-dimensional subspaces of t, where the edges denote containment. The reader may calculate that this graph is an $O\left(\sqrt{\frac{1}{q^{t-1}}}\right)$-bipartite expander. One is often interested in agreement theorems on the Grassmann poset where the local functions are promised to come from some error correcting code. In this case the ensemble f will be a δ-ensemble for constant δ, and therefore we bound the surprise by $\xi(X) \leqslant O(1/q^{t-1})$.

Proof of Lemma II.23. It suffices to show that $\mathbb{P}\left[f_{s_1}\restriction_a = f_{s_2}\restriction_a \mid f_{s_1}\restriction_t \neq f_{s_2}\restriction_t\right] = O\left(\frac{\eta^2}{\delta}\right)$.

Denote by $B = \{v \in t \mid f_{s_1}(v) \neq f_{s_2}(v)\}$. By our assumption on the distance, we are promised that $\mathbb{P}[B] \geqslant \delta$. And indeed, we can invoke the sampler lemma, Lemma A.9, and get that the probability of a to see no vertices in B is $O(\frac{\eta^2}{\delta})$. ◻

D. Main theorem: agreement on STAV structures

We are now ready to state our main technical theorem. Recall that for a given distribution $\mathcal{D}$ over pairs s_1, s_2 we denoted by $\mathrm{rej}_{\mathcal{D}}(f)$ the probability that $f_{s_1}\restriction_t \neq f_{s_2}\restriction_t$ when choosing $s_1, s_2 \sim \mathcal{D}$ and setting $t = s_1 \cap s_2$. For a given STAV X we extend this notation to $\mathrm{rej}_X(f)$ understanding that the sets s_1, t, s_2 are now chosen via the STS distribution that comes with X.

Theorem II.26 (STAV Agreement Theorem)**.** *Let Σ be some finite alphabet (for example $\Sigma = \{0,1\}$). Let $X = (S, T, A, V)$ be a γ-good STAV-structure for some $\gamma < \frac{1}{3}$. Let $f = \{f_s : s \to \Sigma \mid s \in S\}$ be an ensemble such that*

1) *Agreement:*

$$rej_X(f) \leqslant \varepsilon, \tag{II.1}$$

2) *Surprise:*

$$\xi(X, f) \leqslant O(\gamma) \tag{II.2}$$

*Then assuming either Assumption for $r = 1$ or Assumption **(A4)**,*

$$\mathrm{dist}_\gamma(f, \mathcal{G}) \leqslant O(\varepsilon).$$

More explicitly, there exists a global function $G : V \to \Sigma$ s.t.

$$\mathop{\mathbb{P}}_{s\in S}\left[f_s \overset{\gamma}{\neq} G\restriction_s\right] \overset{\mathrm{def}}{=} \mathop{\mathbb{P}}_{s\in S}\left[\mathop{\mathbb{P}}_{v\in V}\left[f_s(v) \neq G\restriction_s \mid v \in s\right] \geqslant \gamma\right] = O(\varepsilon). \tag{II.3}$$

*Moreover, for any $r > 0$, if either Assumption (A4(r))or Assumption **(A4)** holds then*

$$\mathop{\mathbb{P}}_{s\in S}\left[f_s \overset{r\gamma}{\neq} G\restriction_s\right] = O\left(\left(1 + \frac{1}{r}\right)\varepsilon\right). \tag{II.4}$$

The O notation does not *depend on any parameter including γ, ε, the size of the alphabet, the size of $|S|, |T|, |A|, |V|$ and, size of any $s \in S$.*

III. Proof of Main Theorem

In this section we prove our main theorem, Theorem II.26.

We first give a direct proof for the case of two-sided high dimensional expanders, that follows the same line of general proof. Afterwards we prove the theorem in full generality.

A. Proof for a Representative Case: Two-Sided High Dimensional Expanders

In this section we give a direct proof to a special case of our main theorem. We give a sound agreement test on set systems coming from a two-sided high dimensional expander.

We recall that a simplicial complex X is a family of subsets that is downwards closed to containment, i.e. if $s \in X$ and $t \subset s$ the then $t \in X$. We denote by $X(\ell)$ all subsets (also called faces) of size $\ell + 1$. We identify $X(0)$ with the set of vertices. A complex is d-dimensional if the largest faces have size $d + 1$. Our test is the following:

Definition III.1 (d, ℓ-agreement distribution)**.** Let X be a d-dimensional simplicial complex and $\ell < d$ be a positive integer. We define the distribution $D_{d,\ell}$ by the following random process

1) Sample $t \in X(\ell)$.
2) Sample $s_1, s_2 \in X(d)$ independently, given that $t \subset s_1, s_2$.

The d,ℓ-agreement test is the test associated with the d,ℓ-agreement distribution on this family.

Theorem III.2 (Agreement for High Dimensional Expanders)**.** *There exists a constant $c > 0$ such that for every $d > 0$ such that the following holds. Suppose that X is a $\frac{1}{d^3}$-two-sided d-dimensional HDX, and $\ell = \lfloor \frac{d}{3} \rfloor$. Then the d,ℓ-agreement test is exactly c-sound.*

This theorem holds for a wider range of parameters. Also, in this section we will assume that the alphabet is binary, namely that the local functions are $f_s : s \to \{0,1\}$. The full theorem, is discussed and proven in the full paper.

1) Proof of Theorem III.2: The proof of the theorem goes through some auxiliary functions:

Definition III.3 (local popularity function)**.** For every $a \in X(\ell - 1)$ define $h_a : a \to \Sigma$ by popularity, i.e. $h_a = \text{pop}_{s\supset a}\{f_s{\restriction}_a\}$. The notation pop refers to the value $f_s{\restriction}_a$ with highest probability over $s \supset a$, ties are broken arbitrarily.

Definition III.4 (the reach function)**.** For every $a \in X(\ell - 1)$ define $g_a : X_a(0) \to \Sigma$ by the popularity conditioned on $f_s{\restriction}_a = h_a$, i.e.

$$g_a(v) = \text{pop}\{f_s(v) : s \supset a, f_s{\restriction}_a = h_a\}.$$

Ties are broken arbitrarily.

First, we will prove the following lemma on the local popularity functions:

Lemma III.5. *For any $a \in X(\ell - 1)$, let h_a be as in Definition III.3. Denote by ε_a the disagreement probability given that the intersection $t \in X(\ell)$ contains a. That is,*

$$\varepsilon_a = \mathbb{P}\left[f_{s_1}{\restriction}_t \neq f_{s_2}{\restriction}_t \mid\ a \subset t\right].$$

Then for every $a \in X(\ell - 1)$:

$$\mathop{\mathbb{P}}_{s\in X(d)}\left[f_s{\restriction}_a \neq h_a \mid\ s \supset a\right] = O(\varepsilon_a).$$

Next, we move towards showing that when $f_s{\restriction}_a = h_a$, then for a typical a, $f_s(v) = g_a(v)$ occurs with probability $1 - O\left(\frac{\varepsilon}{d}\right)$.

Consider the distribution $(a, s, a') \sim D_{comp}$, where we choose $s \in X(d)$ and then two $a, a' \subset s$ uniformly at random given that they are disjoint.

We say that a triple (a, s, a') is *bad* if $f_s{\restriction}_a \neq h_a$ or $f_s{\restriction}_{a'} \neq h_{a'}$. It is easy to see from Lemma III.17 that there are $O(\varepsilon)$ bad triples at most.

We use the bad triples to define the set of globally bad elements in $X(\ell - 1)$. These are all $a \in X(\ell - 1)$ with many bad triples touching them

$$A^* = \Big\{a \in X(\ell - 1) \Big| \mathop{\mathbb{P}}_{(s,a_2)}\left[(a, s, a_2) \text{ is a bad triple}\right] \geqslant \frac{1}{40}\Big\}. \tag{III.1}$$

We shall use this set A^* to filter and disregard certain $a \in X(\ell - 1)$, that ruin the probability to agree with the $\{g_a\}_{a\in X(\ell-1)}$, and later on with the global function. The constant $\frac{1}{40}$ is arbitrary, and once it is fixed, we can say that $\mathbb{P}[A^*] = O(\varepsilon)$ by Markov's inequality.

Lemma III.6 (agreement with link function)**.** *Let $(a, s, v) \sim D$ be the distribution where we choose $s \in X(d)$ and from it a, v uniformly at random so that $v \notin a$. Then*

$$\mathop{\mathbb{P}}_{(a,v,s)\sim D}\Big[f_s(v) \neq g_a(v) \text{ and } f_s{\restriction}_a = h_a \text{ and } a \notin A^*\Big] = O\left(\frac{\varepsilon}{d}\right). \tag{III.2}$$

Finally our goal is to stitch the g_a's functions together to one global function. Lemma III.18 motivates us to define the global function as the popularity vote on $g_a(v)$ for all $a \in X_v(\ell - 1)$ that see few bad triples when conditioned on v. However, in order to properly define the global function, we need to define another process that takes into account the agreement of two functions $g_a, g_{a'}$. For this we need to look at each vertex $v \in X(0)$ separately.

To do so, we define the following graph:

Definition III.7 (Local Complement Graph)**.** Fix any $v_0 \in X(0)$. The local complement graph H_{v_0} is the graph whose vertices are $V = X_{v_0}(\ell - 1)$. Our labeled edges are chosen as follows: Given that we are at element a we traverse to a' via edge s, by choosing some $s \supset a \cup \{v_0\}$ and then choosing some $a' \subset s$ given that $a \cap a' = \emptyset$.

For $v \in V$, we say $a \in X_v(\ell - 1)$ is *locally bad for* v, if

$$\mathop{\mathbb{P}}_{(a_1,s,a_2)\in E(H_v)} [(a_1, s, a_2) \text{ is bad} \mid \ a_1 = a] > \frac{1}{20}.$$

The constant here is also arbitrary.

Finally, for every $v \in V$, we define A_v^* to be the set of all $a \in X(\ell - 1)$ that are either globally bad, or locally bad for v.

We show using the sampler lemma, Lemma A.9, that if $a \in X(\ell - 1)$ is not globally bad, then the probability over $v \in V$, that it will be locally bad for v is small, i.e.

Claim III.8 (Not Globally Bad implies Not Locally Bad).

$$\mathop{\mathbb{P}}_{a\in X(\ell-1),v\in X_a(0)} [a \in A_v^* \text{ and } a \notin A^*] = O\left(\frac{\varepsilon}{d}\right).$$

Now we can define our global function $G : V \to \Sigma$ as follows:

$$G(v) = \text{pop}\,\{g_a(v) \mid \ a \in X_v(\ell-1), a \notin A_v^*\},$$

as usual, ties are broken arbitrarily. In words, we remove a small amount of bad $a \in X(\ell - 1)$, where many functions f_s's don't agree with the g_a's, and take the popular vote of the remainder.

Using the local complement graph and Claim III.8, we can now prove:

Lemma III.9 (agreement with global function)**.**

$$\mathop{\mathbb{P}}_{a\in X(\ell-1),v\in X_0(a)} [g_a(v) \neq G(v) \textit{ and } a \notin A_v^*] = O\left(\frac{\varepsilon}{d}\right).$$

Given the lemmata above, we prove the theorem.

Proof of Theorem III.2. We note that it is enough to show

$$\mathop{\mathbb{P}}_{s\in X(d),a\in X(\ell-1),a\subset s} \left[f_s{\restriction_{s\setminus a}} \neq G{\restriction_{s\setminus a}}\right] = O(\varepsilon). \quad \text{(III.3)}$$

This is due to the fact that $|s \setminus a| \geqslant \frac{1}{2}|s|$, thus if $f_s \neq G{\restriction_s}$, then $f_s{\restriction_{s\setminus a}} \neq G{\restriction_{s\setminus a}}$ for at least half of the possible $a \subset s$.

Next, we prove (III.3). We define the following events, when we choose (a, s, v) in the simplicial complex:

1) E_1 - the event that $f_s{\restriction_a} \neq h_a$.
2) E_2 - the event that $a \in A^*$, i.e. the a chosen has many bad edges.

Define a random variable Z, that samples s, a and outputs

$$Z(s,a) = \mathop{\mathbb{P}}_{v\in s\setminus a} [f_s(v) \neq G(v)], \quad \text{(III.4)}$$

i.e. the fraction of vertices in $s \setminus a$ so that $f_s(v) \neq G(v)$.

The probability for $E_1 \vee E_2$ is $O(\varepsilon)$ by Lemma III.5 and Markov's inequality.

If $\neg(E_1 \vee E_2)$, yet a vertex v contributes to the probability in (III.4), then one of the three must occur:

1) $a \in A_v^*$.
2) $f_s(v) \neq g_a(v)$ and $a \notin A_v^*$.
3) $a \notin A_v^*$ but $f_s(v) = g_a(v) \neq G(v)$.

The first event occurs with probability $O\left(\frac{\varepsilon}{d}\right)$ by Claim III.8. The second occurs with probability $O\left(\frac{\varepsilon}{d}\right)$ by Lemma III.6. The third occurs with probability $O\left(\frac{\varepsilon}{d}\right)$ by Lemma III.9. Thus by the expectation of Z given that $\neg(E_1 \vee E_2)$ is $O\left(\frac{\varepsilon}{d}\right)$. By Markov's inequality

$$\mathop{\mathbb{P}}_{s\in X(d),a\in X(\ell-1),a\subset s} \left[f_s{\restriction_{s\setminus a}} \neq G{\restriction_{s\setminus a}} \middle| \neg(E_1 \vee E_2)\right] = \mathbb{P}\left[Z \geqslant \frac{1}{d} \middle| \ \neg(E_1 \vee E_2)\right] = |s \setminus a| O\left(\frac{\varepsilon}{d}\right) = O(\varepsilon).$$

In conclusion

$$\mathop{\mathbb{P}}_{s\in X(d)} [f_s \neq G{\restriction_s}] \leqslant \mathbb{P}[E_1 \vee E_2] + \mathop{\mathbb{P}}_{s\in X(d),a\in X(\ell-1),a\subset s} \left[f_s{\restriction_{s\setminus a}} \neq G{\restriction_{s\setminus a}} \middle| \ \neg(E_1 \vee E_2)\right] = O(\varepsilon).$$

□

2) Proof of the Lemmata:

Lemma (Restatement of Lemma III.5)**.** *For any $a \in X(\ell-1)$, let h_a be as in Definition III.3. Denote by ε_a the disagreement probability given that the intersection $t \in X(\ell)$ contains a. That is,*

$$\varepsilon_a = \mathbb{P}\left[f_{s_1}\restriction_t \neq f_{s_2}\restriction_t \mid \ a \subset t\right].$$

Then for every $a \in X(\ell-1)$:

$$\mathbb{P}_{s \in X(d)}\left[f_s\restriction_a \neq h_a \mid \ s \supset a\right] = O\left(\varepsilon_a\right).$$

Proof of Lemma III.5. Fix $a \in X(\ell-1)$. If $\varepsilon_a \geqslant \frac{1}{6}$ we are trivially done, so assume otherwise. Consider the following graph:

1) The elements in the graph are all $s \supset a$.
2) We connect two elements s_1, s_2 whenever there exists some $t \in X(\ell)$, $t \supset a$ so that $s_1 \cap s_2 \supset t$.

The random walk in this graph is as follows: given s_1 we traverse to s_2 by the d,ℓ-agreement test's distribution, that is, given that the intersection contains a.

We denote the walk we just described, by the d,ℓ-containment walk. In [18], the authors proved the following theorem, regarding this walk:

Theorem III.10. *Let X be a λ-one sided link expander. Then the d,ℓ-containment walk is $\frac{\ell+1}{d+1} + O\left(d\lambda\right)$-two-sided spectral expander.*

This theorem is thoroughly discussed in the full version of this paper.

By Theorem III.10, this graph is a very good spectral expander. In particular, it is a $\frac{1}{3}$-edge expander, when d is sufficiently large.

We color the vertices of this graph according to their value at a. Denote by $S_1, S_2, \ldots$ the colors, where S_1 is the largest. Namely, S_1 are all the s so that $f_s\restriction_a = h_a$.

Denote by $S_i = \{s : f_s\restriction_a = h_a^i\}$. We need to show that the set of vertices $S_1 = \{s : f_s\restriction_a = h_a\}$ (the largest of all S_i) is $1 - O\left(\varepsilon_a\right)$.

The quantity ε_a, i.e. the amount of edges between S_i's, is by assumption less than $\frac{1}{6}$.

By Claim A.6, using the fact that the graph is a $\frac{1}{3}$-edge expander and the fact that the fraction of edges between the S_i's is less that $\frac{1}{6}$. We get that $\mathbb{P}\left[S_1\right] \geqslant \frac{1}{2}$.

Furthermore, by the edge-expander property $\mathbb{P}\left[S_1^c\right] \leqslant 3E(S_1, S_1^c) \leqslant 3\varepsilon_a$. □

Corollary III.11. $\mathbb{P}\left[A^*\right] = O\left(\varepsilon\right)$.

Proof of Corollary III.11. Each $a \in X(\ell-1)$ contributes to A^* if the amount of bad triples that a participates in is $\geqslant \frac{1}{40}$. The total amount of bad triples is $O\left(\varepsilon\right)$ by Lemma III.5. Thus by Markov's inequality $\mathbb{P}\left[A\right] = O\left(\varepsilon\right)$. □

We move to Claim III.8.

Claim (Restatement of Claim III.8).

$$\mathbb{P}_{a \in X(\ell-1), v \in X_a(0)}\left[a \in A_v^* \text{ and } a \notin A^*\right] = O\left(\frac{\varepsilon}{d}\right).$$

Proof of Claim III.8. Fix some $a \notin A^*$. Consider the following bipartite graph:

- $L = \{(a', s) : a' \cup a \subset s\}$.
- $R = X_a(0)$.
- $E = \{(v, (a', s)) : \{v\} \cup a' \cup a \subset s\}$,

The probability to choose each edge is given by the following distribution in the link X_a):

1) Sample $v \in X_a(0)$.
2) Sample $s \setminus a \in X_a(d-\ell)$ so that $v \in s$.
3) Sample $a' \in X_a(\ell-1)$ so that $a' \subset s \setminus \{v\}$.

Note that the probability of (a', s) in the left side, is precisely the probability to choose the triple $(a, s, a') \sim D_{comp}$, given that the first element is a.

Denote by $B \subset L$ the that consists of all (s, a') s.t. (a, s, a') is a bad triple. If $a \notin A^*$ then $\mathbb{P}\left[B\right] < \frac{1}{40}$.

In the full version of the paper, we give an analysis of this graph's random walk. In particular we prove the following statement:

Proposition III.12. *Fix some $a \in A$, and consider the following graph*

- *$L = \{(a', s) : a' \cup a \subset s\}$.*
- *$R =$ $reach_a = X_a(0)$.*
- *$E = \{(v, (a', s)) : \{v\} \cup a' \cup a \subset s\}$, and the probability to choose each edge is given by the distribution that chooses (s, a', v) in the link of a.*

The graph described above is an $O\left(\frac{1}{\sqrt{\ell}}\right)$-bipartite expander.

By Proposition III.12, this graph is a $O\left(\frac{1}{\sqrt{d}}\right)$-bipartite expander.

Define the set

$$V^* = \left\{ v \in \text{ reach}_a \;\middle|\; \mathop{\mathbb{P}}_{(s,a')}\left[B \;\middle|\; v \sim (s,a')\right] \right.$$

the set of $v \in$ reach$_a$ so that the probability for a bad edge is larger than $\frac{1}{20}$, namely, that a is locally bad for v.

In the sampler lemma, Lemma A.9, we see that bipartite-expanders are good samplers. We use Lemma A.9 to get that $\mathbb{P}[V^*] = O\left(\frac{1}{d}\right)\mathbb{P}[B]$. Taking expectation on all $a \in A$ we get that

$$\begin{aligned}
&\mathop{\mathbb{P}}_{a\in X(\ell-1), v\in X_a(0)}[a \in A_v^* \text{ and } a \notin A^*] = \\
&\mathbb{P}[a \notin A^*] \cdot \mathop{\mathbb{E}}_{a\notin A^*}[\mathbb{P}[V^*]] = \\
&\mathbb{P}[a \notin A^*] \cdot O\left(\frac{1}{d}\right) \mathop{\mathbb{E}}_{a\notin A^*}[\mathbb{P}[B]] \leqslant \\
&O\left(\frac{1}{d}\right) \mathop{\mathbb{E}}_{a\in A}[\mathbb{P}[B]] = O\left(\frac{\varepsilon}{d}\right),
\end{aligned}$$

The last inequality is due to the fact that taking expectation on this set conditioned on $a \notin A^*$, is less than the expectation on all A (by definition when $a \in A^*$, then $\mathbb{P}[B] \geqslant \frac{1}{40}$, and when $a \notin A^*$, $\mathbb{P}[B] < \frac{1}{40}$). The last equality is since $\mathbb{P}[B] = O(\varepsilon)$ by Corollary III.11. □

We move towards proving Lemma III.6. We shall use the following "surprise" claim.

Claim III.13 (Surprise). Let $\hat{D}$ denote the distribution where we sample:

1) $a \in X(\ell-1)$.
2) $v \in X_a(0)$.
3) $s_1, s_2 \in X(d)$ independently, given that they contain $t = a \cup \{v\}$.

Then

$$\mathop{\mathbb{P}}_{\hat{D}}[f_{s_1}(v) \neq f_{s_2}(v) \text{ and } f_{s_1}\restriction_a = f_{s_2}\restriction_a] = O\left(\frac{\varepsilon}{d}\right).$$

This claim is given in full generality in that is given in the full paper. For this section to be self contained, we give it an elementary proof:

Proof of Claim III.13. $\hat{D}$ can be described as first choosing s_1, s_2, t and then partitioning $t = a \cup \{v\}$. So from the law of total probability we obtain:

$$\mathop{\mathbb{P}}_{\hat{D}}[f_{s_1}(v) \neq f_{s_2}(v) \text{ and } f_{s_1}\restriction_a = f_{s_2}\restriction_a] =$$

$$\left.\geqslant \frac{1}{20}\right\}, \quad \mathop{\mathbb{E}}_{(t,s_1,s_2)}\left[\mathop{\mathbb{P}}_{v\in t, a=t\setminus\{v\}}[f_{s_1}(v) \neq f_{s_2}(v) \text{ and } f_{s_1}\restriction_a = f_{s_2}\restriction_a]\right].$$

Notice that for every $t \in X(\ell)$, the $\{s_1, s_2\}$ pairs that contribute to the probability above, are the ones that fail the test (but do so on exactly one vertex). By the agreement test, there are at most an ε-fraction of such pairs. Given that we choose such a pair, their contribution to the expectation is $\frac{1}{\ell} = O\left(\frac{1}{d}\right)$ since that is the probability of choosing the $v \in t$ s.t. $f_{s_1}(v) \neq f_{s_2}(v)$. □

Now we are ready to prove Lemma III.6.

Lemma (Restatement of Lemma III.6)**.** *Let $(a,s,v) \sim D$ be the distribution where we choose $s \in X(d)$ and from it a, v uniformly at random so that $v \notin a$. Then*

$$\mathop{\mathbb{P}}_{(a,v,s)\sim D}\left[f_s(v) \neq g_a(v) \text{ and } f_s\restriction_a = h_a \text{ and } a \notin A^*\right] = O\left(\frac{\varepsilon}{d}\right).$$

The proof we give here relies on the fact that the alphabet is binary, or at least of size $O(1)$. It is possible to prove this for an alphabet of unbounded size, as we do in the main proof.

Proof of Lemma III.6. First, note that by Claim III.8, (III.2) is less or equal to

$$\begin{aligned}
&\mathbb{P}[a \notin A^* \text{ and } a \in A_v^*] + \\
&\mathop{\mathbb{P}}_{(a,v,s)\sim D}[f_s(v) \neq g_a(v) \text{ and } f_s\restriction_a = h_a \text{ and } a \notin A_v^*] = \\
&O\left(\frac{\varepsilon}{d}\right) + \mathop{\mathbb{P}}_{(a,v,s)\sim D}[f_s(v) \neq g_a(v) \text{ and } f_s\restriction_a = h_a \text{ and } a \notin A_v^*].
\end{aligned}$$

Thus we focus on bounding

$$\mathop{\mathbb{P}}_{(a,v,s)\sim D}[f_s(v) \neq g_a(v) \text{ and } f_s\restriction_a = h_a \text{ and } a \notin A_v^*]. \tag{III.5}$$

We write the expression we want to bound in (III.5) as

$$\mathbb{E}_{a,v}\left[\mathbb{P}_s\left[f_s(v) \neq g_a(v) \text{ and } f_s\restriction_a \neq h_a \text{ and } a \notin A_v^*\right]\right].$$

We denote the expression inside the expectation as $p_{a,v} = \mathbb{P}_s\left[f_s(v) \neq g_a(v) \text{ and } f_s\restriction_a \neq h_a \text{ and } a \notin A_v^*\right]$. Thus we want to show that

$$\mathbb{E}_{a,v}\left[p_{a,v}\right] = O\left(\frac{\varepsilon}{d}\right).$$

By Claim III.13, we got that

$$\mathbb{P}_{(a,v,s_1,s_2)\sim\hat{D}}\left[f_{s_1}(v) \neq f_{s_2}(v) \text{ and } f_{s_1}\restriction_a = f_{s_2}\restriction_a\right] = O\left(\frac{\varepsilon}{d}\right).$$

We can write this also as an expectation over a, v:

$$\mathbb{E}_{a,v}\left[\mathbb{P}_{(s_1,s_2)}\left[f_{s_1}(v) \neq f_{s_2}(v) \text{ and } f_{s_1}\restriction_a = f_{s_2}\restriction_a\right]\right] = O\left(\frac{\varepsilon}{d}\right).$$

We denote the expression in the expectation by $,q_{a\ v} = \mathbb{P}_{(s_1,s_2)}\left[f_{s_1}(v) \neq f_{s_2}(v) \text{ and } f_{s_1}\restriction_a = f_{s_2}\restriction_a\right]$.

Our goal is to relate the two quantities, namely, to show that $p_{a,v} = O\left(q_{a,v}\right)$. This will show that

$$\mathbb{E}_{a,v}\left[p_{a,v}\right] = O\left(\mathbb{E}_{a,v}\left[q_{a,v}\right]\right) = O\left(\frac{\varepsilon}{d}\right).$$

Fix some $a \in X(\ell-1)$ and $v \in X_a(0)$. If $a \in A_v^*$ then $p_{a,v} = 0$ and we are done. So assume $a \notin A_v^*$.

Denote by H_0 the set of all $s \supset t = a \cup \{v\}$. In the sampling process for $p_{a,v}$ we choose some $s \in H_0$, and in the sampling process for $q_{a,v}$ we choose $s_1, s_2 \in H_0$ independently.

We can partition H_0 to

$$H_0 = G \cup B,$$

where G contains all $s \in H_0$ so that $f_s\restriction_a = h_a$. B is all $s \in H_0$ so that $f_s\restriction_a \neq h_a$.

$a \notin A_v^*$, thus

$$\mathbb{P}_{s\in H_0}[B] < \frac{1}{20},$$

or

$$\mathbb{P}_{s\in H_0}[G] > \frac{19}{20}.$$

Thus, conditioning on G doesn't change the probability of $q_{a,v}$ significantly, namely

$$\mathbb{P}_{s_1,s_2}\left[f_{s_1}\restriction_a = f_{s_2}\restriction_a \text{ and } f_{s_1}(v) \neq f_{s_2}(v) \middle| s_1, s_2 \in G\right] \leqslant 2q_{a,v}.$$

The first equality in the probability, $f_{s_1}\restriction_a = f_{s_2}\restriction_a$, is immediately satisfied in this set, since if $s_1, s_2 \in G$ then $f_{s_1}\restriction_a = h_a = f_{s_2}\restriction_a$. So we get

$$\mathbb{P}_{s_1,s_2}\left[f_{s_1}(v) \neq f_{s_2}(v) \mid\ s_1, s_2 \in G\right] \leqslant 2q_{a,v}.$$

Because s_1, s_2 are chosen independently, we can say that

$$\mathbb{P}_{s_1,s_2}\left[f_{s_1}(v) \neq f_{s_2}(v) \mid\ s_1, s_2 \in G\right] = 2\,\mathbb{P}_{s_1}\left[f_{s_1}(v) = g_a(v) \mid\ s_1 \in G\right] \cdot \mathbb{P}_{s_2}\left[f_{s_2}(v) \neq g_a(v) \mid\ s_2 \in G\right].$$ [3]

The definition of $g_a(v)$ is taking the majority of $f_s(v)$ for all $s \in G$. Thus $\mathbb{P}_{s_1}\left[f_{s_1}(v) = g_a(v) \mid\ s_1 \in G\right] \geqslant \frac{1}{2}$.

$$\mathbb{P}_{s_1,s_2}\left[f_{s_1}(v) \neq f_{s_2}(v) \mid\ s_1, s_2 \in G\right] \geqslant \mathbb{P}_{s_2}\left[f_{s_2}(v) \neq g_a(v) \mid\ s_2 \in G\right] \geqslant p_{a,v}.$$

The last inequality is by the definition of $p_{a,v}$. In conclusion, $p_{a,v} \leqslant 2q_{a,v}$ and we are done. □

We state this immediate corollary:

Corollary III.14. *Consider the following distribution $(v, a, s, a') \sim D_{vasa}$, where (a, s, a') are chosen by D_{comp} and v is sampled from $s \setminus (a \cup a')$ uniformly at random. Then*

$$\mathbb{P}_{(v,a,s,a')\sim D_{vasa}}\left[f_s\restriction_{a_i} = h_{a_i} \textit{ and } g_{a_1} \neq g_{a_2} \textit{ and } a_i \notin A_v^* \textit{ for } i = 1, 2\right] = O\left(\frac{\varepsilon}{d}\right).$$

□

The proof for this corollary is just applying Lemma III.6 for each a_i and using a union bound.

It remains to prove Lemma III.9.

Lemma (Restatement of Lemma III.9)**.**

$$\mathbb{P}_{a\in X(\ell-1),v\in X_0(a)}\left[g_a(v) \neq G(v) \textit{ and}\right.$$

$$a \notin A_v^*]$$
$$= O\left(\frac{\varepsilon}{d}\right).$$

For the proof of the lemma, we'll need the following property of expander graphs. In an expander graph, the number of outgoing edges from some $A \subset V$, is an approximation to the size of A or $V \setminus A$. Claim A.10 generalizes this fact to the setting where we count only outgoing edges from A to a (large) set $B \subset V \setminus A$.

Proof of Lemma III.9. Fix some $v_0 \in X(0)$. We view the local complement graph H_0 from Definition III.7.

The walk on this graph is the $\ell-1, \ell-1$-complement walk in the link of v. By Theorem A.21,that we prove in the full paper, this graph is a $O\left(\frac{1}{d}\right)$-two-sided spectral expander.

Consider the following sets in this graph: $M_{v_0} = \left\{a \in X_{v_0}(\ell-1) \setminus A_{v_0}^* \;\middle|\; g_a(v_0) = G(v)\right\}$ the popular vote, $N_{v_0} = \left\{a \in X_{v_0}(\ell-1) \setminus A_{v_0}^* \;\middle|\; g_a(v_0) \neq G(v)\right\}$ the other votes, and $C_{v_0} = A_{v_0}^*$. The $a \in N_{v_0}$ are those where $g_a(v_0) \neq G(v_0)$ and $a \notin A_{v_0}^*$. Hence we need to bound

$$\mathop{\mathbb{E}}_{v_0}\left[\mathbb{P}\left[N_{v_0}\right]\right].$$

We invoke Claim A.10 for $N_{v_0}, M_{v_0}, C_{v_0}$ and get that

$$\mathbb{P}\left[N_{v_0}\right] \leqslant \frac{1}{\left(1 - O\left(\frac{1}{d}\right)\right)\mathbb{P}\left[M_{v_0}\right]}\mathbb{P}\left[E(N_{v_0}, M_{v_0})\right] + O\left(\frac{1}{d}\right)\mathbb{P}\left[C_{v_0}\right]. \quad \text{(III.6)}$$

The proof now has two steps:

1) We show that $\mathbb{P}\left[M_{v_0}\right] \geqslant \frac{9}{20}$ for all but $O\left(\frac{\varepsilon}{d}\right)$ of the vertices v_0 (the constant is arbitrary). This will imply that the denominator in (III.6) is larger than some constant (say $\frac{1}{2}$).
2) We show that the right hand side of (III.6) is $O\left(\frac{\varepsilon}{d}\right)$ in expectation.

a) To show step 1.: We will need to show that for all but $O\left(\frac{\varepsilon}{d}\right)$ of the v_0, the size of C_{v_0} is smaller than $\frac{1}{20}$, namely

$$\mathop{\mathbb{P}}_{v}\left[\mathbb{P}\left[A_v^*\right] > \frac{1}{20}\right] = O\left(\frac{\varepsilon}{d}\right) \quad \text{(III.7)}$$

Assuming that for $\mathbb{P}\left[C_{v_0}\right] \leqslant \frac{1}{20}$, it is obvious that $\mathbb{P}\left[M_{v_0}\right] \geqslant \frac{9}{20}$, using the fact that the alphabet is binary in this special case, thus M_{v_0} is the larger set between M_{v_0}, N_{v_0}.

To show (III.7) consider the complement graph between $X(0)$ and $X(\ell-1)$, where the edges are all (v,a) so that $\{v\} \cup a \in X(\ell)$. This is the $0,(\ell-1)$-complement walk.

The set of vertices v that we need to bound is the set of v's with large $\mathbb{P}\left[A_v^*\right] > \frac{1}{20}$. There are two types of vertices v:

- $\mathbb{P}\left[A_v^* \cap A^*\right] \leqslant \frac{1}{40}$
- $\mathbb{P}\left[A_v^* \cap A^*\right] > \frac{1}{40}$

By Claim III.8, $\mathbb{P}_{(a,v)}\left[a \in A_v^* \text{ and } a \notin A^*\right] = O\left(\frac{\varepsilon}{d}\right)$. Thus by Markov's inequality, only $O\left(\frac{\varepsilon}{d}\right)$ of the vertices can see $\frac{1}{40}$ fraction of neighbors $a \in A_v^* \setminus A^*$, thus bounding by $O\left(\frac{\varepsilon}{d}\right)$ the fraction of v's of the first type.

To bound the vertices of the second type, note that these are vertices that have a large ($> \frac{1}{40}$) fraction of neighbors in A^*. By Corollary III.11, $\mathbb{P}\left[A^*\right] = O\left(\varepsilon\right)$. According to Theorem A.21, our graph is a $\sqrt{\frac{1}{d}}$-bipartite expander. Thus by the sampler lemma Lemma A.9, the set of vertices $v_0 \in X(0)$ who have more than $\frac{1}{40}$-fraction neighbours in A^*, is $O\left(\frac{\varepsilon}{d}\right)$.

b) As for step 2.: Taking expectation on (III.6) we have that

$$\begin{aligned}
&\mathbb{E}[\mathbb{P}\left[N_{v_0}\right]] \leqslant \\
&\mathbb{E}\left[\frac{1}{\left(1 - O\left(\frac{1}{d}\right)\right)\mathbb{P}\left[M_{v_0}\right]}\mathbb{P}\left[E(N_{v_0}, M_{v_0})\right]\right] \\
&\qquad + O\left(\frac{1}{d}\right)\mathbb{E}[\mathbb{P}\left[C_{v_0}\right]] \\
&\leqslant \mathop{\mathbb{P}}_{v}\left[\mathbb{P}\left[A_v^*\right] > \frac{1}{20}\right] \\
&\quad + \mathbb{E}[4\,\mathbb{P}\left[E(N_{v_0}, M_{v_0})\right]] \\
&\qquad + O\left(\frac{1}{d}\right)\mathbb{E}[\mathbb{P}\left[C_{v_0}\right]], \quad \text{(III.8)}
\end{aligned}$$

where the second inequality is due to the fact that when $\mathbb{P}_v\left[\mathbb{P}\left[A_v^*\right] \leqslant \frac{1}{20}\right]$ then

$$\frac{1}{\left(1 - O\left(\frac{1}{d}\right)\right)\mathbb{P}\left[M_{v_0}\right]} \leqslant 4.$$

We bound each of the terms on the right hand side of (III.8) separately.

By (III.7),

$$\mathop{\mathbb{P}}_{v}\left[\mathbb{P}\left[A_v^*\right] > \frac{1}{20}\right] = O\left(\frac{\varepsilon}{d}\right).$$

By Corollary III.11 and Claim III.8

$$O\left(\frac{1}{d}\right)\mathop{\mathbb{E}}_{v_0}\left[\mathbb{P}\left[C_{v_0}\right]\right] = O\left(\frac{1}{d}\right)\mathop{\mathbb{E}}_{v}\left[\mathbb{P}\left[A_v^*\right]\right] = O\left(\frac{\varepsilon}{d}\right).$$

We continue to bound $\mathbb{P}\left[E(N_{v_0}, M_{v_0})\right]$ in expectation. Every edge counted in $E(N_{v_0}, M_{v_0})$ is either a bad triple (i.e. and edge (a_1, s, a_2) s.t. $f_s\restriction_{a_i} \neq h_{a_i}$ for $i = 1$ or 2), or a non-bad edge (an edge who is not bad) for which we see a disagreement. By Corollary III.14 there are $O\left(\frac{\varepsilon}{d}\right)$ non-bad edges in the cut.

As for the bad edges, notice that $a \in N_{v_0}$ is not a member of $A_{v_0}^*$, thus the amount of bad edges that are connected to a is at most $\frac{1}{20}$-fraction of the edges connected to a (by definition). Thus the amount of bad edges is bounded by $\frac{1}{20}\,\mathbb{P}\left[N_{v_0}\right]$, and

$$\mathbb{P}\left[E(N_{v_0}, M_{v_0})\right] \leqslant O\left(\frac{\varepsilon}{d}\right) + \frac{1}{20}\,\mathbb{P}\left[N_{v_0}\right].$$

By summing up the bounds we get that

$$\mathbb{E}[\mathbb{P}\left[N_{v_0}\right]] \leqslant O\left(\frac{\varepsilon}{d}\right) + \frac{4}{20}\,\mathbb{E}[\mathbb{P}\left[N_{v_0}\right]]$$

hence

$$\mathbb{E}[\mathbb{P}\left[N_{v_0}\right]] = O\left(\frac{\varepsilon}{d}\right).$$

□

B. *Proof for the General Case*

Next we prove Theorem II.26 in full generality.

The proof of the theorem goes through these auxiliary functions:

Definition III.15 (local popularity function)**.** For every $a \in A$ define $h_a : a \to \Sigma$ by popularity, i.e. $h_a = \mathrm{pop}_{s \supset a} f_s\restriction_a$. The notation pop refers to the value $f_s\restriction_a$ with highest probability over $s \supset a$, ties are broken arbitrarily.

Definition III.16 (the reach function)**.** For every $a \in A$ define $g_a :\ \mathrm{reach}_a \to \Sigma$ by the popularity conditioned on $f_s\restriction_a = h_a$, i.e.

$$g_a(v) = \mathrm{pop}\{f_s(v) : a \subset s, f_s\restriction_a = h_a\}.$$

Ties are broken arbitrarily.

First, We will prove the following lemma on the local popularity functions:

Lemma III.17. *For any $a \in A$, let h_a be as in Definition III.15. Denote by ε_a the disagreement probability given that $t' \supset a$, i.e.*

$$\varepsilon_a = \mathbb{P}\left[f_{s_1}\restriction_{t'} \neq f_{s_2}\restriction_{t'} \;\middle|\; t' \in \{t \supset a\}\right].$$

Then for every $a \in A$:

$$\mathop{\mathbb{P}}_{s \in \{s \supset a\}}\left[f_s\restriction_a \neq h_a\right] = O\left(\varepsilon_a\right).$$

Next, we move towards showing that when $f_s\restriction_a = h_a$, then for a typical a, $f_s(v) = g_a(v)$ occurs with probability $1 - O\left(\gamma\varepsilon\right)$.

We consider the $VASA$-distribution. We say that a triple (a, s, a') is *bad* if $f_s\restriction_a \neq h_a$ or $f_s\restriction_{a'} \neq h_{a'}$, in the context of the ${}_VASA$-graphs defined in Section II-B, we call these triples bad edges, since the edges of the ${}_VASA$-graphs correspond to triples (a, s, a'). It is easy to see from Lemma III.17 that there are $O\left(\varepsilon\right)$ bad edges at most.

We use the bad triples to define the set of globally bad elements in A, to be all $a \in A$ with many bad triples touching them

$$A^* = \left\{a \in A \middle| \mathop{\mathbb{P}}_{(s,a_2)}\left[(a, s, a_2) \text{ is a bad edge}\right] \geqslant \frac{1}{40}\right\}$$

namely, all the $a \in A$ so that the probability in Lemma III.17 given that we chose a fixed $a \in A$, is larger than a constant. We shall use this set A^* to filter and disregard certain $a \in A$, that ruin the probability to agree with the $\{g_a\}_{a \in A}$, and later on with the global function. The constant $\frac{1}{40}$ is arbitrary, and once it is fixed, we can say that $\mathbb{P}\left[A^*\right] = O\left(\varepsilon\right)$ by Markov's inequality.

Lemma III.18 (agreement with link function)**.** *Let D be a distribution over $(a, s, v) \in A \times S \times V$ defined by the STAV-structure, that is:*

1) *Choose some (a, v) where $v \in \mathrm{reach}_a$.*
2) *Choose some $(a, v) \subset s$ (where we mean $\{v\}, a \subset s$).*

Then

$$\mathop{\mathbb{P}}_{(a,v,s)\sim D}[f_s(v) \neq g_a(v) \textit{ and } f_s\restriction_a = h_a \textit{ and } a \notin A^*_v]$$
$$= O(\gamma\varepsilon). \quad \text{(III.9)}$$

Finally our goal is to stitchg_a's functions together to one global function.

Lemma III.18 motivates us to define the global function as the popularity vote on $g_a(v)$ for all $a \notin A^*_v$ such that $v \in$ reach$_a$. However, in order to properly define the global function, we need to define another process that takes into account the agreement of two functions $g_a, g_{a'}$. For this we use the $_vASA$-graphs described in Assumption **(A3)**a.

For $v \in V$, we say $a \in$ reach$_v$ is *locally bad for* v, if

$$\mathop{\mathbb{P}}_{(a_1,s,a_2)\in E(_vASA)}[(a_1, s, a_2) \text{ is bad} \mid a_1 = a] >^{-} \frac{1}{20}$$

The constant here is also arbitrary.

Finally, for every $v \in V$, we define A^*_v to be the set of all $a \in$ reach$_v$ that are either globally bad, or locally bad for v.

We show using the sampler lemma, Lemma A.9, that if $a \in A$ is not globally bad, then the probability over $v \in V$, that it will be locally bad for v is small, i.e.

Claim III.19 (Not Globally Bad implies Not Locally Bad).

$$\mathop{\mathbb{P}}_{a\in A, v\in \text{ reach}_a}[a \in A^*_v \text{ and } a \notin A^*] = O(\gamma\varepsilon).$$

Now we can define our global function $G : V \to \Sigma$ as follows:

$$G(v) = \text{pop}\{g_a(v) \mid a \in \text{ reach}_v,\ a \notin A^*_v\},$$

as usual, ties are broken arbitrarily. In words, we remove a small amount of bad $a \in A$, where many functions f_s's don't agree with the g_a's, and take the popular vote of the remainder.

We can now prove:

Lemma III.20 (agreement with global function)**.**

$$\mathop{\mathbb{P}}_{a\in A, v\in \textit{ reach}_a}[g_a(v) \neq G(v) \textit{ and } a \notin A^*_v]$$
$$= O(\gamma\varepsilon).$$

Given the lemmata above, we prove the theorem for STAV-structures.

Proof of Theorem II.26. We first show that based on Assumption **(A4)** or Assumption , it is enough to prove

$$\mathop{\mathbb{P}}_{s\in S, a\in s}\left[f_s\restriction_{s\cap\text{ reach}_a} \overset{\frac{1}{2}r\gamma}{\neq} G\restriction_{s\cap\text{ reach}_a}\right]$$
$$= O\left(\left(1+\frac{1}{r}\right)\varepsilon\right). \quad \text{(III.10)}$$

Indeed for any $r > 0$,

– If Assumption **(A4)** holds, and
$$f_s \overset{r\gamma}{\neq} G\restriction_s.$$
it implies that
$$f_s\restriction_{s\cap\text{ reach}_a} \overset{\frac{1}{2}r\gamma}{\neq} G\restriction_{s\cap\text{ reach}_a}$$
for all $a \subset s$. Thus there cannot be more than $O\left(\left(1+\frac{1}{r}\right)\varepsilon\right)$ $s \in S$ as above.
If Assumption (A4(r)) holds for $r\gamma$, then for any
$$f_s\restriction_{s\cap\text{ reach}_a} \overset{r\gamma}{\neq} G\restriction_{s\cap\text{ reach}_a},$$
it is true by the assumption that a $\frac{2}{3}$-fraction of the $a \subset s$ have the property that
$$f_s\restriction_{s\cap\text{ reach}_a} \overset{\frac{1}{3}r\gamma}{\neq} G\restriction_{s\cap\text{ reach}_a}.$$
Hence
$$\mathop{\mathbb{P}}_{s}\left[f_s \overset{r\gamma}{\neq} G\restriction_s\right] \leqslant$$
$$\frac{3}{2}\mathop{\mathbb{P}}_{s\in S, a\in s}\left[f_s\restriction_{s\cap\text{ reach}_a} \overset{\frac{1}{3}r\gamma}{\neq} G\restriction_{s\cap\text{ reach}_a}\right]$$
$$= O\left(\left(1+\frac{1}{r}\right)\varepsilon\right)$$
and we are done.

Next, we prove (III.10). We define the following events:

1) E_1 - the event that $f_s\restriction_a \neq h_a$.
2) E_2 - the event that $a \in A^*$, i.e. the a chosen has many bad edges.

Define a random variable Z, that samples s, a and outputs

$$\mathop{\mathbb{P}}_{v\in s\cap\text{ reach}_a}[f_s(v) \neq G(v)]. \quad \text{(III.11)}$$

The probability for $E_1 \vee E_2$ is $O(\varepsilon)$ by Lemma III.17 and Markov's inequality.

If $\neg(E_1 \vee E_2)$, yet a vertex v contributes to the probability in (III.11), then one of the three must occur:

1) $a \in A^*_v$.
2) $f_s(v) \neq g_a(v)$ and $a \notin A^*$.
3) $a \notin A^*_v$ but $f_s(v) = g_a(v) \neq G(v)$.

The first event occurs with probability $O(\gamma\varepsilon)$ by Claim III.19. The second occurs with probability $O(\gamma\varepsilon)$ by Lemma III.18. The third occurs with probability $O(\gamma\varepsilon)$ by Lemma III.20. Thus by the expectation of Z given that $\neg(E_1 \vee E_2)$ is $O(\gamma\varepsilon)$. By Markov's inequality for any $r > 0$,

$$\mathop{\mathbb{P}}_{s\in S, a\in s}\left[f_s{\restriction}_{s\cap\text{ reach}_a} \overset{r\gamma}{\neq} G{\restriction}_{s\cap\text{ reach}_a} \;\middle|\; \neg(E_1 \vee E_2)\right] = O\left(\frac{\varepsilon}{r}\right).$$

In conclusion

$$\mathop{\mathbb{P}}_{s\in S, a\in s}\left[f_s{\restriction}_{s\cap\text{ reach}_a} \overset{r\gamma}{\neq} G{\restriction}_{s\cap\text{ reach}_a}\right] \leqslant \mathbb{P}\left[E_1 \vee E_2\right] + \mathop{\mathbb{P}}_{s\in S, a\in s}\left[f_s{\restriction}_{s\cap\text{ reach}_a} \overset{r\gamma}{\neq} G{\restriction}_{s\cap\text{ reach}_a} \;\middle|\; \neg(E_1 \vee E_2)\right] = O\left(\left(1+\frac{1}{r}\right)\varepsilon\right).$$

□

C. Proof of the Lemmata

Lemma (Restatement of Lemma III.17)**.** *For any $a \in A$, let h_a be as in Definition III.15. Denote by ε_a the disagreement probability given that $t' \supset a$, i.e.*

$$\varepsilon_a = \mathbb{P}\left[f_{s_1}{\restriction}_{t'} \neq f_{s_2}{\restriction}_{t'} \;\middle|\; t' \in \{t \supset a\}\right].$$

Then for every $a \in A$:

$$\mathop{\mathbb{P}}_{s\in\{s\supset a\}}\left[f_s{\restriction}_a \neq h_a\right] = O\left(\varepsilon_a\right).$$

Proof of Lemma III.17. Fix $a \in A$, and denote by ε_a the probability to succeed in the STS-test given that $s_1, s_2, t \supset a$. If $\varepsilon_a \geqslant \frac{1}{6}$ we are trivially done, so assume otherwise. Denote by $\{h_a^i\}_i$ all possible functions from a to Σ, where $h_a^1 = h_a$. Consider the STS_a-graph. According to Assumption **(A2)**a, this is a $\frac{1}{3}$-edge expander.

Denote by $S_i = \{s : f_s{\restriction}_a = h_a^i\}$. We need to show that the set of vertices $S_1 = \{s : f_s{\restriction}_a = h_a\}$ (the largest of all S_i) is $1 - O(\varepsilon_a)$.

The quantity ε_a, i.e. the amount of edges between S_i's, is by assumption less than $\frac{1}{6}$. The STS_a-graph is a $\frac{1}{3}$-edge expander.

It is a known fact that if we partition a vertex of an edge-expander graph, and there are few outgoing edges, then one of the parts in the partition is large. This fact is formulated in Claim A.6.

We invoke Claim A.6, using the fact that the graph is a $\frac{1}{3}$-edge expander and the fact that the fraction of edges between the S_i's is less that $\frac{1}{6}$. We get that $\mathbb{P}\left[S_1\right] \geqslant \frac{1}{2}$.

By the edge-expander property $\mathbb{P}\left[S_1^c\right] \leqslant 3E(S_1, S_1^c) \leqslant 3\varepsilon_a$.

□

Corollary III.21. $\mathbb{P}\left[A^*\right] = O\left(\varepsilon\right)$.

Proof of Corollary III.21. $a \in A$ contributes to A^* if the amount of bad edges that a participates in is $\geqslant \frac{1}{40}$. The total amount of bad edges is $O(\varepsilon)$ by Lemma III.17. Thus by Markov's inequality $\mathbb{P}\left[A\right] = O\left(\varepsilon\right)$. □

We move to Claim III.19.

Claim (Restatement of Claim III.19)*.*

$$\mathop{\mathbb{P}}_{a\in A, v\in\text{ reach}_a}\left[a \in A_v^* \text{ and } a \notin A^*\right] = O\left(\gamma\varepsilon\right).$$

Proof of Claim III.19. Fix some $a \notin A^*$. Consider the VAS_a-graph for this a. This is the bipartite graph, where $L = \text{reach}_{a_0}$, $R = \{(a, s) \mid \exists v \in L\ (v, a_0, s, a) \in \mathit{Supp}(D)\}$, $E = \{(v, (a, s)) : (v, a_0, s, a) \in \mathit{Supp}(D)\}$. The probability of choosing an edge $(v, (a', s))$ is given by $\mathbb{P}_D\left[(v, a_0, s, a') \mid a_0 = a\right]$.

Denote by $B \subset L$ the that consists of all (s, a') s.t. (a, s, a') is bad. If $a \notin A^*$ then $\mathbb{P}\left[B\right] < \frac{1}{40}$. From Assumption **(A3)**b, this graph is a $\sqrt{\gamma}$-bipartite expander. Define the set

$$V^* = \left\{v \in \text{ reach}_a \;\middle|\; \mathop{\mathbb{P}}_{(s,a')}\left[B \mid v \sim (s, a')\right] \geqslant \frac{1}{20}\right\},$$

the set of $v \in \text{ reach}_a$ so that the probability for a bad edge is larger than $\frac{1}{20}$, namely, that a is locally bad for v. In the sampler lemma, Lemma A.9, we see that bipartite-expanders are good samplers.

We use Lemma A.9 to get that $\mathbb{P}\left[V^*\right] = O\left(\gamma\right)\mathbb{P}\left[B\right]$. Taking expectation on all $a \in A$ we get that

$$\mathop{\mathbb{P}}_{a\in A, v\in\text{ reach}_a}\left[a \in A_v^* \text{ and } a \notin A^*\right] = \mathbb{P}\left[a \notin A^*\right] \cdot \mathop{\mathbb{E}}_{a\notin A^*}\left[\mathbb{P}\left[V^*\right]\right] =$$

$$\mathbb{P}\left[a \notin A^*\right] \cdot O\left(\gamma\right) \mathop{\mathbb{E}}_{a \notin A^*}\left[\mathbb{P}\left[B\right]\right] \leqslant$$
$$O\left(\gamma\right) \mathop{\mathbb{E}}_{a \in A}\left[\mathbb{P}\left[B\right]\right] = O\left(\gamma\varepsilon\right),$$

The last inequality is due to the fact that taking expectation on this set conditioned on $a \notin A^*$, is less than the expectation on all A (by definition when $a \in A^*$, then $\mathbb{P}\left[B\right] \geqslant \frac{1}{40}$, and when $a \notin A^*$, $\mathbb{P}\left[B\right] < \frac{1}{40}$). The last equality is since $\mathbb{P}\left[B\right] = O\left(\varepsilon\right)$ by Lemma III.17. □

Moving on to Lemma III.18:

Lemma (Restatement of Lemma III.18)**.** *Let D be a distribution over $(a, s, v) \in A \times S \times V$ defined by the STAV-structure, that is:*

1) *Choose some (a, v) where $v \in reach_a$.*
2) *Choose some $(a, v) \subset s$ (where we mean $\{v\}, a \subset s$).*

Then

$$\mathop{\mathbb{P}}_{(a,v,s)\sim D}\left[f_s(v) \neq g_a(v) \text{ and } f_s{\restriction_a} = h_a \text{ and } a \notin A_v^*\right] = O\left(\gamma\varepsilon\right).$$

For the proof of the lemma, we'll need the following property of expander graphs. In an expander graph, the number of outgoing edges from some $A \subset V$, is an approximation to the size of A or $V \setminus A$. Claim A.10 generalizes this fact to the setting where we count only outgoing edges from A to a (large) set $B \subset V \setminus A$.

Proof of Lemma III.18. For a fixed (a_0, v_0) we consider he conditioned STS_{a_0,v_0}-graph. Recall that the vertices in this graph are all the $s \supset (a, v)$ and the edges are (s, t, s') so that $t \supset (a, v)$.

We partition this graph to three sets: $M_{a_0,v_0} = \left\{s \in V \middle|\ f_s{\restriction_{a_0}} = h_{a_0}, f_s(v_0) = g_a(v_0)\right\}$, $N_{a_0,v_0} = \left\{s \in V \middle|\ f_s{\restriction_{a_0}} = h_{a_0}, f_s(v_0) \neq g_a(v_0)\right\}$, $C_{a_0,v_0} = \left\{s \in V \middle|\ f_s{\restriction_{a_0}} \neq h_{a_0}\right\}$. For (a_0, v_0) so that $a_0 \notin A_{v_0}^*$, the elements $s \in N_{a_0,v_0}$ are exactly those who contribute to the probability in (III.9). Thus the probability in (III.9) is

$$\mathop{\mathbb{P}}_{(a_0,v_0)}\left[a_0 \notin A_{v_0}^*\right] \mathop{\mathbb{E}}_{(a_0,v_0):\, a_0 \notin A_{v_0}^*}\left[\mathbb{P}\left[N_{a_0,v_0}\right]\right].$$

We also denote by H_{a_0,v_0} the set of edges (s_1, t, s_2) in the STS_{a_0,v_0}-graph, so that

$$f_{s_1}(v_0) \neq f_{s_2}(v_0) \text{ and } f_{s_1}{\restriction_{a_0}} = f_{s_2}{\restriction_{a_0}}.$$

Note that any edge between N_{a_0,v_0} and M_{a_0,v_0} is an edge of H_{a_0,v_0}. By (II.2), $\xi(f) = \gamma$. Thus in particular

$$\begin{aligned}\mathop{\mathbb{P}}_{(s_1,s_2,t,a,v)}&\left[f_{s_1}(v) \neq f_{s_2}(v) \text{ and } f_{s_1}{\restriction_a} = f_{s_2}{\restriction_a}\right] \leqslant \\ \mathop{\mathbb{P}}_{(s_1,s_2,t,a,v)}&\left[f_{s_1}{\restriction_t} \neq f_{s_2}{\restriction_t} \text{ and } f_{s_1}{\restriction_a} = f_{s_2}{\restriction_a}\right] \\ &= \mathbb{P}\left[f_{s_1}{\restriction_t} \neq f_{s_2}{\restriction_t}\right]\xi(f) \\ &= \gamma\varepsilon.\end{aligned}$$

Thus

$$\mathop{\mathbb{E}}_{(a_0,v_0)}\left[\mathbb{P}\left[H_{a_0,v_0}\right]\right] = \mathbb{P}\left[f_{s_1}(v) \neq f_{s_2}(v) \text{ and } f_{s_1}{\restriction_a} = f_{s_2}{\restriction_a}\right] = O\left(\gamma\varepsilon\right).$$

According to Assumption **(A2)**b, the STS_{a_0,v_0}-graph is a γ-two-sided spectral expander, thus we can use the almost cut approximation property Claim A.10 to show that

$$(1-\gamma)\,\mathbb{P}\left[M_{a_0,v_0}\right]\mathbb{P}\left[N_{a_0,v_0}\right] = O\left(\mathbb{P}\left[E(N_{a_0,v_0}, M_{a_0,v_0})\right] + \gamma\,\mathbb{P}\left[C_{a_0,v_0}\right]\right). \tag{III.12}$$

To conclude the proof we show first that the right hand side of (III.12) is bound by $O\left(\gamma\varepsilon\right)$ in expectation over (a_0, v_0). Then we show that for all but $O\left(\gamma\varepsilon\right)$ of the (a_0, v_0),

$$\mathbb{P}\left[M_{a_0,v_0}\right] \geqslant \frac{1}{2}. \tag{III.13}$$

Indeed, as

$$E(N_{a_0,v_0}, M_{a_0,v_0}) \subset H_{a_0,v_0},$$

we get that

$$\mathop{\mathbb{P}}_{(a_0,v_0)}\left[a_0 \notin A_{v_0}^*\right] \cdot \mathop{\mathbb{E}}_{(a_0,v_0):\, a_0 \notin A_{v_0}^*}\left[\mathbb{P}\left[E(N_{a_0,v_0}, M_{a_0,v_0})\right]\right] \leqslant \mathop{\mathbb{E}}_{(a_0,v_0)}\left[\mathbb{P}\left[H_{a_0,v_0}\right]\right] = O\left(\gamma\varepsilon\right).$$

Furthermore, Note that

$$\mathop{\mathbb{P}}_{(a_0,v_0)}\left[a_0 \notin A_{v_0}^*\right] \mathop{\mathbb{E}}_{(a_0,v_0):\, a_0 \notin A_{v_0}^*}\left[\gamma\,\mathbb{P}\left[C_{a_0,v_0}\right]\right] \leqslant$$

$$\gamma \mathop{\mathbb{E}}_{(a_0,v_0)} \left[\mathbb{P}\left[C_{a_0,v_0}\right]\right] = \gamma \, \mathbb{P}\left[f_s\restriction_a \neq h_a\right].$$

This is bounded by $O(\gamma\varepsilon)$ by Lemma III.17.

Hence the right hand side of (III.12) is bound by $O(\gamma\varepsilon)$ in expectation over (a_0, v_0).

To complete the proof, we turn to showing (III.13) for all but $O(\gamma\varepsilon)$ of the (a_0, v_0). For this, we use the edge expander partition property, Claim A.6.

Partition the vertices of the STS_{a_0,v_0}-graph to $B_1, B_2, ...B_{n+1}$ where $B_1 = M_{a_0,v_0}, B_2 = C_{a_0,v_0}$ and $N_{a_0,v_0} = B_3 \cup ... \cup B_n$, where each B_j is the set of s so that $f_s(v) = \sigma_j$ for all $\sigma_j \in \Sigma$.

We assumed that $\mathbb{P}\left[C_{v_0}\right] \leqslant \frac{1}{20}$ hence $E(B_2, B_2^c) = E(C, C^c) \leqslant \frac{1}{20}$.

From (II.2), $\mathbb{E}_{(a_0,v_0}\left[\mathbb{P}\left[H_{a_0,v_0}\right]\right] = O(\gamma\varepsilon)$. From Markov's inequality, $\mathbb{P}\left[H_{a_0,v_0}\right] < \frac{1}{20}$, for all but $O(\gamma\varepsilon)$ of the (a_0, v_0). When this occurs, the amount of edges between the partition parts is $\frac{3}{20} < \frac{1}{6}$.

From the edge expander partition property Claim A.6 we get that one of the partition sets has probability $\geqslant \frac{1}{2}$. This is not $B_2 = C$, as its probability is $\leqslant \frac{1}{20}$. Thus $\mathbb{P}\left[M_{a_0,v_0}\right] \geqslant \frac{1}{2}$.

Thus by using the fact that $\gamma < \frac{1}{3}$, for all but $O(\gamma\varepsilon)$ of the (a_0, v_0),

$$\mathbb{P}\left[N_{a_0,v_0}\right] = O\left(\mathbb{P}\left[E(N_{a_0,v_0}, M_{a_0,v_0})\right] + \gamma\, \mathbb{P}\left[C_{a_0,v_0}\right]\right).$$

Hence

$$\mathop{\mathbb{E}}_{(a_0,v_0)} \left[\mathbb{P}\left[N_{a_0,v_0}\right]\right] = O(\gamma\varepsilon).$$

□

Corollary III.22. *Consider the $VASA$-distribution promised for us in Assumption **(A3)**.*

$$\mathop{\mathbb{P}}_{(v,a_1,s,a_2)} \left[f_s\restriction_{a_i} = h_{a_i} \text{ and } g_{a_1}(v) \neq g_{a_2}(v) \text{ and } a_i \notin A_v^* \text{ for } i = 1, 2\right] = O(\gamma\varepsilon).$$

Proof of Corollary III.22. The probability is bounded by twice the probability we bound in Lemma III.18, and the probability we bound in Claim III.19. □

1) Proof of Lemma III.20: We restate Lemma III.20:

Lemma (Restatement of Lemma III.20)**.**

$$\mathop{\mathbb{P}}_{a\in A, v\in \text{ reach}_a} \left[g_a(v) \neq G(v) \text{ and } a \notin A_v^*\right] = O(\gamma\varepsilon).$$

Proof of Lemma III.20. Fix some $v_0 \in V$ and consider its $_vASA$-graph defined in Section II-B, namely the graph whose vertices reach_{v_0} and edges are all the (a_1, s, a_2) so that (v_0, a_1, s, a_2) is in the support of our $VASA$-distribution.

Consider the following sets in this graph: $M_{v_0} = \left\{a \in \text{ reach}_{v_0} \setminus A_{v_0}^* \,\middle|\, g_a(v_0) = G(v)\right\}$ the popular vote, $N_{v_0} = \left\{a \in \text{ reach}_{v_0} \setminus A_{v_0}^* \,\middle|\, g_a(v_0) \neq G(v)\right\}$ the other votes, and $C_{v_0} = A_{v_0}^*$. The $a \in N$ are those where $g_a(v_0) \neq G(v_0)$ and $a \notin A_{v_0}^*$. Hence we need to bound

$$\mathop{\mathbb{E}}_{v_0} \left[\mathbb{P}\left[N_{v_0}\right]\right].$$

By Assumption **(A3)**a it is a either a γ-bipartite expander or a γ-two-sided spectral expander. Claim A.11, the bipartite almost cut approximation property, is the analogue claim to Claim A.10 for bipartite graphs. We invoke either Claim A.11 or Claim A.10 for $N_{v_0}, M_{v_0}, C_{v_0}$ and get that

$$(1 - 2\gamma)\, \mathbb{P}\left[M_{v_0}\right] \mathbb{P}\left[N_{v_0}\right] \leqslant \mathbb{P}\left[E(N_{v_0}, M_{v_0})\right] + 4\gamma\, \mathbb{P}\left[C_{v_0}\right],$$

or

$$\mathbb{P}\left[N_{v_0}\right] \leqslant \frac{1}{(1 - 2\gamma)\, \mathbb{P}\left[M_{v_0}\right]} \mathbb{P}\left[E(N_{v_0}, M_{v_0})\right] + 4\gamma\, \mathbb{P}\left[C_{v_0}\right]. \quad \text{(III.14)}$$

The proof now has two steps:

1) We show that $\mathbb{P}\left[M_{v_0}\right] \geqslant \frac{1}{2}$ for all but $O(\gamma\varepsilon)$ of the vertices v_0.
2) We show that the right hand side of (III.14) is $O(\gamma\varepsilon)$.

a) To show step 1.: we will need to show that for all but $O(\gamma\varepsilon)$ of the v_0, the size of C_{v_0} is smaller than $\frac{1}{20}$.

$$\mathop{\mathbb{P}}_{v} \left[\mathbb{P}\left[A_v^*\right] > \frac{1}{20}\right] = O(\gamma\varepsilon) \quad \text{(III.15)}$$

Assuming that for $\mathbb{P}[C_{v_0}] \leqslant \frac{1}{20}$, we show that $\mathbb{P}[M_{v_0}] \geqslant \frac{1}{2}$ using the edge expander partition property, Claim A.6.

By Assumption **(A3)**a, the ${}_{v_0}ASA$-graph is a either γ-bipartite expander or a γ-two-sided spectral expander for $\gamma < \frac{1}{3}$, thus it is also a $\frac{1}{3}$-edge expander. We indend to invoke Claim A.6. Partition V to:

- $B_0 = A^*_{v_0} = C_{v_0}$.
- $B_1 = M_{v_0}$.
- $B_2, ..., B_n$ - elements $a \in A$ s.t. $g_a(v) = \sigma_i$ for all $\sigma_i \in \Sigma$ that are not the majority assumption. Note that $A_{v_0} = B_2 \uplus ... \uplus B_n$.

By (III.15), the set $B_0 = A^*_{v_0}$ is $\leqslant \frac{1}{20}$ for all but $O(\gamma\varepsilon)$ of the v's. When this occurs, then $E(C, C^c) \leqslant \frac{1}{10}$.

We bound the amount of edges between the B_i's that are not B_0. We can divide the edges to bad edges, and edges that are not bad.

The "bad edges" between the B_i's account for at most $\frac{1}{20}$ as for every i and every $a \in B_i$, the amount of bad edges connected to it is $\leqslant \frac{1}{20}$ (since $a \notin A^*_{v_0}$).

Finally, from Corollary III.22 and Markov's inequality, there are at most $O(\gamma\varepsilon)$ of the v's where the amount of edges between different B_i's that are not bad is greater than $\frac{1}{20}$.

Thus for all but $O(\gamma\varepsilon)$ of the v's, the amount of edges between parts of this partition is $\leqslant \frac{2}{20} < \frac{1}{6}$. Invoke Claim A.6, to get that one set above must be of size at least $\frac{1}{2}$. This must be $B_1 = M_{v_0}$, as it is larger than the other B_i's where $i \geqslant 1$, and since $B_0 = C_{v_0}$ is of size $\leqslant \frac{1}{20}$.

We move to show that (III.15) is true for all but $O(\gamma\varepsilon)$ of the vertices $v_0 \in V$. Consider the graph between STAV-parts V and A where we choose a pair (a, v) according to the probability to chose them in the $STAV$-structure.

The set of vertices v that we need to bound is the set of v's with large $\mathbb{P}[A^*_v] > \frac{1}{20}$. There are two types of vertices v:

- $\mathbb{P}[A^*_v \cap A^*] \leqslant \frac{1}{40}$
- $\mathbb{P}[A^*_v \cap A^*] > \frac{1}{40}$

By Claim III.19, $\mathbb{P}_{(a,v)}[a \in A^*_v \text{ and } a \notin A^*] = O(\gamma\varepsilon)$. Thus by Markov's inequality, only $O(\gamma\varepsilon)$ of the vertices can see $\frac{1}{40}$-fraction of neighbors $a \in A^*_v \setminus A^*$, thus bounding by $O(\gamma\varepsilon)$ the fraction of v's of the first type.

To bound the vertices of the second type, note that these are vertices that have a large ($> \frac{1}{40}$) fraction of neighbors in A^*. By Corollary III.21, $\mathbb{P}[A^*] = O(\varepsilon)$. According to Assumption **(A1)**, our graph is a $\sqrt{\gamma}$-bipartite expander. Thus by the sampler lemma Lemma A.9, the set of vertices $v_0 \in X(0)$ who have more than $\frac{1}{40}$-fraction neighbours in A^*, is $O(\gamma\varepsilon)$.

b) As for step 2.: Taking expectation on (III.14) we get that

$$\begin{aligned}
&\mathbb{E}[\mathbb{P}[N_{v_0}]] \leqslant \\
&\mathbb{E}[\frac{1}{(1-2\gamma)\,\mathbb{P}[M_{v_0}]}\,\mathbb{P}[E(N_{v_0}, M_{v_0})]] \\
&\qquad + 4\gamma\,\mathbb{E}[\mathbb{P}[C_{v_0}]] \leqslant \\
&\qquad \mathbb{P}_v\left[\mathbb{P}[A^*_v] > \frac{1}{20}\right] \\
&\qquad + \mathbb{E}[6\,\mathbb{P}[E(N_{v_0}, M_{v_0})]] \\
&\qquad\qquad + 4\gamma\,\mathbb{E}[\mathbb{P}[C_{v_0}]], \quad \text{(III.16)}
\end{aligned}$$

where the second inequality is due to the fact that we assumed that $\gamma < \frac{1}{3}$ and that when $\mathbb{P}_v\left[\mathbb{P}[A^*_v] \leqslant \frac{1}{20}\right]$ then $\mathbb{P}[M_{v_0}] \geqslant \frac{1}{2}$, hence

$$\frac{1}{(1-2\gamma)\,\mathbb{P}[M_{v_0}]} \leqslant 6.$$

We bound each of the terms on the right hand side of (III.16) separately.

By (III.15),

$$\mathbb{P}_v\left[\mathbb{P}[A^*_v] > \frac{1}{20}\right] = O(\gamma\varepsilon).$$

By Corollary III.21 and Claim III.19

$$4\gamma\,\mathbb{E}_{v_0}[\mathbb{P}[C_{v_0}]] = 4\gamma\,\mathbb{E}_v[\mathbb{P}[A^*_v]] = O(\gamma\varepsilon).$$

We continue to bound $\mathbb{P}[E(N_{v_0}, M_{v_0})]$ in expectation. Every edge counted in $E(N_{v_0}, M_{v_0})$ is either a bad triple (i.e. and edge (a_1, s, a_2) s.t. $f_s\restriction_{a_i} \neq h_{a_i}$ for $i = 1$ or 2), or a non-bad edge (an edge who is not bad) for which we see a disagreement. By Corollary III.14 there are $O\left(\frac{\varepsilon}{d}\right)$ non-bad edges in the cut.

As for the bad edges, notice that $a \in N_{v_0}$ is not a member of $A^*_{v_0}$, thus the amount of bad edges that are connected to a is at most $\frac{1}{20}$-fraction of the edges connected to a (by definition). Thus the amount of bad edges is bounded by $\frac{1}{20}\,\mathbb{P}\left[N_{v_0}\right]$, and

$$\mathbb{P}\left[E(N_{v_0}, M_{v_0})\right] \leqslant O\left(\gamma\varepsilon\right) + \frac{1}{20}\,\mathbb{P}\left[N_{v_0}\right].$$

By summing up the bounds we get that

$$\mathbb{E}[\mathbb{P}\left[N_{v_0}\right]] \leqslant O\left(\gamma\varepsilon\right) + \frac{6}{20}\,\mathbb{E}[\mathbb{P}\left[N_{v_0}\right]]$$

hence

$$\mathbb{E}[\mathbb{P}\left[N_{v_0}\right]] = O\left(\gamma\varepsilon\right).$$

□

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Appendix

In this appendix we give the necessary background and conventions we use throughout the paper. Most results and claims in this section are standard, and thus given without proof.

A. Expander graphs

Every weighted undirected graph induces a random walk on its vertices: Let $G = (V, E)$ be a finite weighted graph with a probability weight function $\mu : E \to [0, 1]$. The transition probability from v to u is

$$\frac{\mu(\{u, v\})}{\sum_{w\sim v} \mu(\{v, w\})}.$$

Denote by $A = A(G)$ the Markov operator associated with this random walk. We call this operator the *adjacency operator*.

A is an operator on real valued functions on the vertices, where

$$\forall v \in V \ Af(v) = \mathop{\mathbb{E}}_{u\sim v} [f(u)] .$$

The expectation is taken with respect to the graph's probability on vertices, conditioned on being adjacent to v.

A's eigenvalues are in the interval $[-1, 1]$. We denote its eigenvalues by $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$ (with multiplicities). The largest eigenvalue is always $\lambda_1 = 1$, and it is obtained by the constant function. The second eigenvalue is strictly less than 1 if and only if the graph is connected.

Definition A.1 (spectral expanders)**.** Let G be a graph. G is a λ-*one sided spectral expander* for some $0 \leqslant \lambda < 1$, if

$$\lambda_2 \leqslant \lambda.$$

G is a λ-*two sided spectral expander* for some $0 \leqslant \lambda < 1$, if

$$\max(|\lambda_2|, |\lambda_n|) \leqslant \lambda.$$

There is another notion of graph expansion that we'll need in this paper, called edge expansion. Intuitively, an edge expander is a graph where every set of vertices has a large number of outgoing edges.

Definition A.2 (edge expansion)**.** Let G be a weighted graph. The *edge expansion* of G is $\Phi(G) = \min\left\{\frac{\mathbb{P}[E(S,V\setminus S)]}{\mathbb{P}[S]} \,\middle|\, S \subset V,\ 0 < \mathbb{P}[S] \leqslant \frac{1}{2}\right\}$, where $E(S, V \setminus S)$ is the set of all edges between S and $V \setminus S$.

There is a connection between spectral expansion and edge expansion:

Theorem A.3 (Cheeger's inequality)**.** *Let G be any weighted graph. Then*

$$\frac{1 - \lambda_2}{2} \leqslant \Phi(G) \leqslant \sqrt{2(1 - \lambda_2)}.$$

□

1) Bipartite Graphs and Bipartite Expanders: A bipartite graph is a graph where the vertex set can be partitioned to two independent sets $V = L \cup R$, called sides. Bipartite graphs are sometimes easier to analyze than graphs, and arise naturally when studying STAV-structures.

a) The Bipartite Adjacency Operator.: In a bipartite graph, we view each side as a separate probability space, where for any $v \in L$ (resp. R), $\mathbb{P}[v] = \sum_{w \sim v} \mu(\{v, w\})$. We can define the *bipartite adjacency operator* as the operator $B : \ell_2(L) \to \ell_2(R)$ by

$$\forall f \in \ell_2(L), v \in R,\ Bf(v) = \mathop{\mathbb{E}}_{w \sim v}[f(u)]$$

where the expectation is taken with respect to the probability space L, conditioned on being adjacent to v.

We denote by $\lambda(B)$ the spectral norm of B when restricted to $\ell_2^0(L) = \{\mathbf{1}\}^{\perp}$, the orthogonal complement of the constant functions (according to the inner product the measure induces). Namely

$$\lambda(B) = \sup\{\langle Bf, g\rangle \mid \ \|g\|, \|f\| = 1\}.$$

Definition A.4 (Bipartite Expander)**.** Let G be a bipartite graph, let $\lambda < 1$. We say G is a λ-*bipartite expander*, if $\lambda(B) \leqslant \lambda$.

b) Sampling Graph.: We also define a sampling graph, a notion close in some sense to expanders.

Definition A.5 (Sampling Graph)**.** Let $G = (L, R, E)$ be a bipartite graph, and $\delta < 1$. We say that G has the δ-*sampling property* if the following holds: For any set $B \subset V$ of size greater than $\mathbb{P}[C] \geqslant \delta$, the set $T = \{a : \mathbb{P}_{v \in V}[v \in C \mid \ v \in \text{reach}_a] \geqslant \frac{1}{3}\delta\}$ has size at least $\frac{1}{3}$.

B. Properties of Expander Graphs

In this subsection we develop the necessary properties of expander graphs, that we will need in Section III.

a) Edge-Expander Partition Property.: The following claim is also useful in the proof of the main theorem. It says that if we partition the vertices, and there are few edges between the partition's parts, then one set in the partition is larger than $\frac{1}{2}$.

Claim A.6 (Edge-Expander Partition Property). Let $G = (V, E)$ be a c-edge expander. Let $V = B_1 \cup \ldots \cup B_n$, partitioned into sets, and suppose that there are less than $\frac{c}{2}$ edges between parts of the partition, namely:

$$\frac{1}{2}\sum_{i=1}^{n} \mathbb{P}[E(B_i, B_i^c)] < \frac{c}{2}.$$

Then there exists i such that $\mathbb{P}[B_i] \geqslant \frac{1}{2}$.

Proof of Claim A.6. Assume towards contradiction that for all $1 \leqslant i \leqslant n$, $\mathbb{P}[B_i] < \frac{1}{2}$.

From our assumption, there are less than $\frac{c}{2}$ edges between parts of the partition, namely

$$\frac{c}{2} > \frac{1}{2}\sum_{i=1}^{n} \mathbb{P}[E(B_i, B_i^c)] \geqslant \frac{c}{2}\sum_{i=1}^{n} \mathbb{P}[B_i],$$

where the second inequality is from edge expansion. B_i's are a partition of the vertices, thus $\sum_{i=1}^{n} \mathbb{P}[B_i] = 1$, a contradiction.

□

b) Expander Mixing Lemma.: A classical result in expander graphs is the *expander mixing lemma*, that intuitively says that the weight of the edges between any two vertex sets $S, T \subset V$ is proportionate to the probabilities of S, T.

Lemma A.7 (Expander Mixing Lemma)**.** *Let $G = (V, E)$ be a λ-two sided spectral expanders. Then for any $S, T \subset V$*

$$|\mathbb{P}[E(S,T)] - \mathbb{P}[S]\,\mathbb{P}[T]| \leqslant \lambda\sqrt{\mathbb{P}[S]\,\mathbb{P}[T]\,(1 - \mathbb{P}[S])(1 - \mathbb{P}[T])}.$$

□

Bipartite graphs have their own type of expander mixing lemma:

Lemma A.8 (Bipartite Expander Mixing Lemma)**.** *Let $G = (L, R, E)$ be a bipartite λ-one sided spectral expander. Then for any $S \subset L, T \subset R$*

$$|\mathbb{P}[E(S,T)] - \mathop{\mathbb{P}}_{v \in L}[v \in S] \mathop{\mathbb{P}}_{w \in R}[w \in T]| \leqslant \lambda\sqrt{\mathbb{P}[S]\,\mathbb{P}[T]\,(1 - \mathbb{P}[S])(1 - \mathbb{P}[T])}.$$

□

c) Expander Sampler Property.: In [16] the authors showed that bipartite λ-one sided spectral expander has the following useful sampler property.

Lemma A.9 (Sampler Property, by [16])**.** *Let $G = (L, R, U)$ be a bipartite λ-one sided spectral expander. Let $B \subset R$ be any set of vertices, and $c > 0$. then $T = \{v \in L \mid |\mathbb{P}_{w \in R}[w \in S \mid w \sim v] - \mathbb{P}[S]| > c\}$ of vertices who view S as "large", satisfies:*

$$\mathbb{P}[T] \leqslant \frac{\lambda^2}{c^2}\mathbb{P}[S].$$

d) Almost Cut Approximation Property.: As a corollary to the expander mixing lemma, we get the following useful approximation property. In an expander graph, if the number of outgoing edges from some $A \subset V$, is an approximation to the size of A or $V \setminus A$. The following claim generalizes this fact to the setting where we count only outgoing edges from A to a (large) set $B \subset V \setminus A$.

Claim A.10 (Almost Cut Approximation Property). Let $G = (V, E)$ be a λ-two sided spectral expander. Let $V = A \cup B \cup C$, s.t. $\mathbb{P}[A] \leqslant \mathbb{P}[B]$. Then

$$\mathbb{P}[A] \leqslant \frac{1}{(1-\lambda)\,\mathbb{P}[B]}\left(\mathbb{P}[E(A,B)] + \lambda\,\mathbb{P}[C]\right) \tag{A.1}$$

In particular, if $\mathbb{P}[A], 1-\lambda = \Omega(1)$ then

$$\mathbb{P}[A] = O\left(\mathbb{P}[E(A,B)] + \lambda\,\mathbb{P}[C]\right).$$

For bipartite expanders we have an analogues almost approximation cut property, similar to Claim A.10.

Claim A.11 (Almost Cut Approximation Property - Bipartite expanders). Let $G = (L, R, E)$ be a λ-bipartite expander for $\lambda < \frac{1}{2}$. Let $V = A \cup B \cup C$, s.t. $\mathbb{P}[A] \leqslant \mathbb{P}[B]$ (where the probability is taken over all the graph). Then

$$\mathbb{P}[A] \leqslant \frac{1}{2(1-2\lambda)\,\mathbb{P}[B]}\left(\mathbb{P}[E(A,B)] + \lambda 4\,\mathbb{P}[C]\right).$$

In particular, if $\mathbb{P}[A], 1-\lambda = \Omega(1)$ then

$$\mathbb{P}[A] = O\left(\mathbb{P}[E(A,B)] + \lambda\,\mathbb{P}[C]\right).$$

Proofs for Claim A.10 and Claim A.11 are elementary. They can be found in the full version of the paper.

C. Simplicial Complexes and high dimensional expanders

We include here the basic definitions needed for our results. For a more comprehensive introduction to this topic we refer the reader to [16] and the references therein.

A simplicial complex is a hypergraph that is closed downward with respect to containment. It is called d-dimensional if the largest hyperedge has size $d+1$. We refer to $X(\ell)$ as the hyperedges (also called faces) of size $\ell + 1$. $X(0)$ are the vertices.

We define a weighted simplicial complex. Suppose we have a d-dimensional simplicial complex X and a probability distribution $\mu : X(d) \to [0,1]$. We consider the following probabilistic process for choosing lower dimensional faces:

1) Choose some d-face $s_d \in X(d)$ with probability $\mu(s_d)$.
2) Given the choice of s_d, choose sequentially a chain of faces contained in s_d, $(\emptyset \subset s_1 \subset \ldots \subset s_d)$ uniformly, where $s_i \in X(i)$.

For any $s \in X(k)$ we denote by

$$\mathbb{P}[s] = \mathbb{P}\left[\{(\emptyset \subset s_0 \subset \ldots \subset s_d)\} \mid s_k = s\right].$$

For all $s_k \in X(k), s_\ell \in X(l)$, we will write $\mathbb{P}[s_k \mid s_\ell]$ the probability of the k-face in the sequence is s, given that the l-face is s_ℓ.

From here throughout the rest of the paper, when we refer to a simplicial complex X, we always assume that there is a probability measure on it constructed as above.

A link of a face in a simplicial complex, is a generalization of a neighbourhood of a vertex in a graph:

Definition A.12 (link of a face)**.** Let $s \in X(k)$ be some k-face. The *link* of s is a $d-(k+1)$-dimensional simplicial complex defined by:

$$X_s = \{t \backslash s : s \subseteq t \in X\}.$$

The associated probability measure Pr_{X_s}, for the link of s is defined by

$$\mathbb{P}_{X_s}[t] = \mathbb{P}_X[t \cup s \mid s],$$

where Pr_X is the measure defined on X.

Definition A.13 (underlying graph)**.** The *underlying graph* of a simplicial complex X with some probability measure as define above, is the graph whose vertices are $X(0)$ and edges are $X(1)$, with (the restriction of) the probability measures of X to the vertices and edges.

We are ready to define our notion of high dimensional expanders: the one-sided and two-sided link expander.

Definition A.14 (one-sided and two-sided link expander)**.** Let $0 \leqslant \lambda < 1$. A simplicial complex X is a λ*-two sided link expander* (or λ-two sided HDX) if for every $-1 \leqslant k \leqslant d-2$ and every $s \in X(k)$, the underlying graph of the link X_s is a λ-two sided spectral expander.

Similarly, X is a λ*-one sided link expander* (or λ-one sided HDX) if for every $-1 \leqslant k \leqslant d-2$ and every $s \in X(k)$, the underlying graph of the link X_s is a λ-one sided spectral expander.

When X is a graph, this definition coincides with the definition of a spectral expander.

We remark that it is a deep theorem that there exist good one-sided and two-sided high dimensional expanders with bounded degree [30].

$d+1$-partite simplicial complexes: A $d+1$-partite simplicial complex is a generalization of a bipartite graph. We say a d-dimensional simplicial complex is $d+1$*-partite* if we can partition the vertex set

$$V = V_0 \cup V_1 \cup \ldots \cup V_d,$$

s.t. any d-face $s \in X(d)$, contains a vertex from each V_i, i.e. $|s \cap V_i| = 1$.

The *color* of a k-face $s \in X(k)$, is the set of all indexes of V_i's, that intersect with s. I.e.

$$col(s) = \{j \in [d] : |s \cap V_j| = 1\}.$$

For any $J \subset [d]$, we denote

$$X[J] = \{s \in X : col(s) = J\}.$$

When $J = \{i\}$, we abuse the notation and write $X[i]$ instead of $X[\{i\}]$ (not to be confused with $X(i)$).

D. Main Theorem

Theorem A.15 (Restatement of Theorem II.26)**.** *Let Σ be some finite alphabet (for example $\Sigma = \{0,1\}$). Let $X = (S,T,A,V)$ be a γ-good STAV-structure for some $\gamma < \frac{1}{3}$. Let $f = \{f_s : s \to \Sigma \mid s \in S\}$ be an ensemble such that*

1) *Agreement:*

$$rej_X(f) \leqslant \varepsilon, \tag{A.2}$$

2) *Surprise:*

$$\xi(X,f) \leqslant O(\gamma) \tag{A.3}$$

*Then assuming either Assumption for $r=1$ or Assumption **(A4)**,*

$$\mathrm{dist}_\gamma(f,\mathcal{G}) \leqslant O(\varepsilon).$$

More explicitly, there exists a global function $G : V \to \Sigma$ s.t.

$$\mathop{\mathbb{P}}_{s\in S}\left[f_s \overset{\gamma}{\neq} G\restriction_s\right] \overset{\text{def}}{=} \mathop{\mathbb{P}}_{s\in S}\left[\mathop{\mathbb{P}}_{v\in V}\left[f_s(v) \neq G\restriction_s \mid v \in s\right] \geqslant \gamma\right] = O(\varepsilon). \tag{A.4}$$

*Moreover, for any $r > 0$, if either Assumption (A4(r))or Assumption **(A4)** holds then*

$$\mathop{\mathbb{P}}_{s\in S}\left[f_s \overset{r\gamma}{\neq} G\restriction_s\right] = O\left(\left(1+\frac{1}{r}\right)\varepsilon\right). \tag{A.5}$$

The O notation does not *depend on any parameter including γ, ε, the size of the alphabet, the size of $|S|, |T|, |A|, |V|$ and, size of any $s \in S$.*

E. Applications of Main Theorem

1) Agreement tests on two-sided HDX.

Theorem A.16 (Agreement for High Dimensional Expanders)**.** *There exists a constant $c > 0$ such that for every two natural numbers $d > \ell$ such that $\frac{1}{2}d - \ell = \Omega(d)$ the following holds. Suppose that X is a $\frac{1}{d^2\ell}$-two-sided d-dimensional HDX. Then for every $r > 0$ the d,ℓ-agreement test is $\frac{r}{\ell}$-approximately $\left(c(1+\frac{1}{r})\right)$-sound. In particular, if $\ell = \Omega(d)$, then the test is* exactly *c-sound.*

2) Agreement tests on one-sided HDX.

Theorem A.17 (Agreement for $(d+1)$-Partite High Dimensional Expanders)**.** *There exists a constant $c > 0$ such that for every two natural numbers k, ℓ so that $k \geqslant 4\ell + 4$ the following holds. Suppose X is a k-dimensional skeleton of a $(d+1)$-Partite $\frac{1}{k^2\ell}$-one sided HDX (including $k = d$)*[4]*. Then for every $r > 0$ the d, ℓ-agreement test is $\frac{r}{\ell}$-approximately $\left(c\left(1+\frac{1}{r}\right)\right)$-sound. In particular, if $\ell = \Omega(k)$, then the test is exactly c-sound.*

3) Agreement tests on vertex neighbourhoods.

Theorem A.18 (Agreement on neighborhoods)**.** *There exists a constant $c > 0$ such that for every non-negative integers ℓ, k, d such that $4 \leqslant \ell \leqslant \frac{d-2}{2}$ and $\ell + 2k + 2 \leqslant d$, the following holds. Let X be a d-dimensional $\frac{1}{\ell\,(k+\ell)^2}$-two-sided high dimensional expander. Then the ℓ, k-weak independent agreement test (where we choose two links of k-faces that share a common ℓ face), is $\frac{1}{\ell}$-approximately c-sound.*

4) Agreement tests on the Affine and Linear Grassmann Posets:

Theorem A.19 (Agreement on the Affine Grassmann Poset)**.** *There exists a constant $c > 0$ such that for every prime power q, $r, \delta > 0$, and integers ℓ, d, n such that $3\ell + 2 < d \leqslant n$ the following holds. The d, ℓ-Grassmann agreement test* [5] *on $X = Gr_{aff}(\mathbb{F}^n, d)$ (affine spaces of dimension d) is $q^{-\ell}r\delta$-approximately $c\left(1+\frac{1}{r}\right)$-sound for δ-ensembles.*

Theorem A.20 (Agreement on the Linear Grassmann Poset)**.** *There exists a constant $c > 0$ such that for every prime power q, $r, \delta > 0$, and integers ℓ, d, n such that $3\ell + 2 < d \leqslant n$ the following holds. The d, ℓ-Grassmann agreement test on $X = Gr_{lin}(\mathbb{F}^n, d)$ (linear subspaces of dimension d) is $q^{-\ell+1}r\delta$-approximately $c\left(1+\frac{1}{r}\right)$-sound for δ-ensembles.*

[4] a k-skeleton of a d-dimensional simplicial complex Y is $X = \{s \in X \mid \; |s| \leqslant k+1\}$.

[5] That is, sampling two d-dimensional affine spaces that share an ℓ-dimensional affine subspace.

F. Analysis of the Complement Walk

Theorem A.21. *1) Let X be a λ two-sided d-dimensional link-expander. Let ℓ_1, ℓ_2 integers so that $\ell_1 + \ell_2 + 1 \leqslant d$. Denote by M^{ℓ_1,ℓ_2}, the bipartite operator of the ℓ_1, ℓ_2-complement walk. Then $\lambda(M^{\ell_1,\ell_2}) \leqslant (\ell_1+1)(\ell_2+1)\lambda$.*

2) Let X be a $d+1$-partite $\frac{\lambda}{(d+1)\lambda+1}$-one-sided link expander, where $\lambda < \frac{1}{2}$. Let $I, J \subset [d]$ be two disjoint colors. Denote by $M^{I,J}$ the I, J-colored walk. Then $\lambda(M^{I,J}) \leqslant |I||J|\lambda$.

G. High Dimensional Expander Mixing Lemma

1) Two sided case:

Lemma A.22 (High dimensional expander mixing lemma - two-sided)**.** *Let X be a d-dimensional λ-two sided link expander. Let $j_1, j_2, ..., j_m \leqslant d$, and $A_1 \subset X(j_1), A_2 \subset X(j_2), ..., A_m \subset X(j_m)$ s.t. for any $j_{\ell_1} \neq j_{\ell_2}$, and any $s \in A_{j_{\ell_1}}, t \in A_{j_{\ell_2}}$, $s \cap t = \emptyset$. Then*

$$\Bigg| \mathbb{P}\left[F(A_1, A_2, ..., A_k)\right] - \binom{k+1}{j_1+1, j_2+1, ..., j_m+1} \prod_{j=1}^{m} \mathbb{P}\left[A_j\right] \Bigg| \leqslant C\lambda \sqrt[m]{\prod_{j=1}^{m} \mathbb{P}\left[A_j\right]}$$

where C depends on m, d only.[6]

2) One sided partite case:

Lemma A.23 (High dimensional expander mixing lemma - one-sided $d+1$-partite)**.** *Let X be a λ-one sided $d+1$-partite link expander. Let $I_1, ..., I_m \subset [d+1]$ be pairwise disjoint colors, and let $A_1 \subset X[I_1], ..., A_m \subset X[I_m]$. Then*

$$\Bigg| \mathbb{P}\left[F(A_1, A_2, ..., A_k)\right] - \prod_{j=1}^{m} \mathbb{P}\left[A_j \mid \; X[I_j]\right] \Bigg| \leqslant C\lambda \sqrt[m]{\prod_{j=1}^{m} \mathbb{P}\left[A_j \mid \; X[I_j]\right]}$$

[6] here $\binom{k+1}{j_1+1, j_2+1, ..., j_m+1}$ is the number of partitions of a set of size $k+1$ to sets of size $j_1+1, j_2+1, ..., j_m+1$.

where C depends on m, d only.

Name	Definition	Reference
STAV-Structure	A system of sets with four layers: S - sets, T - intersections, A - amplification, V - vertices. It is accompanied by a distribution $(s, t, (a, v)) \sim D_{stav}$.	Definition II.5
STS-distribution	A distribution where we sample $t \in T$, and then $s_1, s_2 \in S$ so that $s_1 \cap s_2 \supset t$. The marginal (s_i, t) is the same as the marginal in D_{stav}.	Definition II.5
VASA-distribution	A distribution $(v, a, s, a') \sim D_{vasa}$ where the marginals $(v, a, s), (v, a', s)$ are the same as D_{stav}.	Definition II.5
Reach Graph	The bipartite graph between V and A where we choose an edge (v, a) according to the STAV-distribution. We denote by reach_a or reach_v then neighbours of a or v in this graph, respectively.	Definition II.9.
Local Reach Graph (AV_s-graph)	For a fixed $s_0 \in S$, the AV_{s_0}-graph is a bipartite graph where $L = \{a \mid a \subset s_0\}$ and $R = \{v \mid v \in s_0\}$. The edges are chosen according to the STAV-distribution given that $s = s_0$.	Definition II.10
STS_a-Graph	For a fixed $a_0 \in A$, the STS_{a_0}-graph is a graph whose elements are $\{s \mid s \supset a_0\}$. We connect s, s' when there exists $t \in T$ so that $a_0 \subset t \subset s \cap s'$.	Definition II.11
$\text{STS}_{a,v}$-Graph	For a fixed $a_0 \in A$ and $v_0 \in \text{reach}_{a_0}$, the STS_{a_0,v_0}-graph is a graph whose elements are $\{s \mid s \supset (a_0, v_0)\}$. We connect s, s' when there exists $t \in T$ so that $(a_0, v_0) \subset t \subset s \cap s'$.	Definition II.12
${}_vASA$-graph	For a fixed $v_0 \in V$ the ${}_{v_0}ASA$-graph is a graph whose elements are $a \in \text{reach}_{v_0}$. We connect a, a' with a labeled edge (a, s, a') if (v_0, a, s, a') is in the support of D_{vasa}.	Definition II.13
Bipartite VAS_a-Graph	For a fixed $a_0 \in A$, the VAS_{a_0}-Graph is a bipartite graph where one side is $L = \text{reach}_{a_0}$. The other side is the set of (s, a') so that (a_0, s, a') is in the support of the marginal of D_{vasa}. We sample an edge in this graph by sampling (v, a, s, a') given that $a = a_0$.	Definition II.14
Surprise	Let $\{f_s\}_{s \in S}$ be some local ensemble. The surprise of the ensemble is the probability over (s, a, v) that $f_s{\restriction}_a = f_{s'}{\restriction}_a$ but $f_s(v) \neq f_{s'}(v)$.	Definition II.17