Fleischer po-semigroups and quantum B-algebras

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***Abstract*—Following the idea of Fleischer who represented BCK-algebras by means of residuable elements of commutative integral po-monoids, we describe quantum B-algebras as subsets of residuable elements of po-semigroups. Moreover, we show that quantum B-algebras correspond one-to-one to what we call Fleischer po-semigroups. Such an approach is more economical than using logical quantales introduced by Rump.**

***Index Terms*—Partially ordered semigroup, residuable element, residuated partially ordered semigroup, quantale, quantum B-algebra, Fleischer po-semigroup, (pseudo-) BCK-algebra.**

INTRODUCTION

Residuation is one of fundamental concepts in partially ordered structures and categories. In the context of logic, residuated lattices and other residuated structures (although they originated in a different context) arise as algebraic models of substructural logics [6].

On the object level, unital quantales are exactly complete residuated lattices. The term *quantale* was suggested by Mulvey [11] as a "quantization" of the term locale. Quantales are applied in linear and other substructural logics or in automata theory [1], [18].

The residuation subreducts of residuated structures give algebraic models of the implicational fragments of the corresponding logics. Perhaps the most general structures of this kind are Rump's *quantum B-algebras* [13], [14], which are partially ordered algebras arising as subreducts of quantales and which subsume most of other implicational algebras such as BCK-algebras and pseudo-BCK-algebras [7] (also known as biresiduation algebras [17]). Therefore, quantum B-algebras (possibly equipped with additional operations) can provide a uniform algebraic semantics for a wide class of substructural logics.

Every quantum B-algebra [13]–[15] (and in particular, every BCK- and pseudo-BCK-algebra [2], [10], [17]) can be embedded into a quantale, but the connection between a given quantum B-algebra and its enveloping quantale may be quite loose. On the other hand, Rump and Yang [13], [15] proved that there is a one-to-one correspondence between quantum B-algebras and the so-called *logical quantales*.

In our paper we follow the idea of Fleischer [2] who represented BCK-algebras using residuable elements of commutative integral partially ordered monoids. Our aim is to represent quantum B-algebras [13], [14] by means of residuable elements of partially ordered semigroups and show that there is a one-to-one correspondence between quantum B-algebras and what we call *Fleischer po-semigroups.*

The paper is organized as follows. First, in Section I we present several necessary algebraic concepts such as a quantum B-algebra, a partially ordered semigroup (po-semigroup) and a residuable element. Then we describe quantum B-algebras as subsets of residuables of partially ordered semigroups.

In Section II we give a representation of quantum B-algebras and related implicational algebras using the notion of a Fleischer po-semigroup. Finally, Section III concludes this paper.

In this paper, we take for granted the concepts and results on quantales and residuated lattices. For more information on these topics we direct the reader to [6], [9] and [12].

I. EVERY QUANTUM B-ALGEBRA IS A SUBSET OF RESIDUABLES IN A PO-SEMIGROUP

In this section, we construct a partially ordered semigroup $\mathcal{M}$ for every quantum B-algebra $\boldsymbol{A}$ such that we can identify $\boldsymbol{A}$ with a subset $\mathcal{A}$ of the set of residuables of $\mathcal{M}$.

First, we recall some basic concepts.

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A *partially ordered semigroup* (or a *po-semigroup* for short) is a structure $\boldsymbol{A} = (A, \leq, \cdot)$ where $\leq$ is a partial order and $\cdot$ is a binary operation that is associative and order-preserving in both variables. If $x \cdot y \leq x$ and $y \cdot x \leq x$ for all $x, y \in A$, then $\boldsymbol{A}$ is called *two-sided* (or *negative* [5]).

A *partially ordered monoid* (or a *po-monoid* for short) is a structure $\boldsymbol{A} = (A, \leq, \cdot, e)$ such that $(A, \leq, \cdot)$ is a po-semigroup and $(A, \cdot, e)$ is a monoid. If the unit e is also the greatest element of the poset $(A, \leq)$, then the po-monoid $\boldsymbol{A}$ is called *integral*.

In a po-semigroup $\boldsymbol{A}$, we call an element $z \in A$:

(i) *weakly residuable* if there are unary operations $\backslash z$ and $z/$ which satisfy

$$x \cdot y \leq z \text{ iff } x \leq z/y \text{ iff } y \leq x\backslash z$$

for all $x, y \in A$; in other words, if for all $x, y \in A$, both $z/y = \max\{a \in A \mid a \cdot y \leq z\}$ and $x\backslash z = \max\{a \in A \mid x \cdot a \leq z\}$ exist;

(ii) *residuable* if z is weakly residuable and, for all $x, y \in A$, the elements z/y and $x\backslash z$ are weakly residuable.

We let $r(\boldsymbol{A})$ and $R(\boldsymbol{A})$ denote the set of weakly residuable elements of $\boldsymbol{A}$ and the set of residuable elements of $\boldsymbol{A}$, respectively.

The following statement is evident.

Lemma 1. *Let $\boldsymbol{A}$ be a po-semigroup.*

(i) *If $z \in r(\boldsymbol{A})$, $a, b \in A$ and $a \leq b$, then $z/b \leq z/a$ and $b\backslash z \leq a\backslash z$.*

(ii) *If $z \in r(\boldsymbol{A})$ and $a, b \in A$, then $z/(a \cdot b) = (z/b)/a$ and $(a \cdot b)\backslash z = b\backslash(a\backslash z)$; if $z \in R(\boldsymbol{A})$, then also $a\backslash(z/b) = (a\backslash z)/b$.*

(iii) *If $\boldsymbol{A}$ is a po-monoid with unit e and $z \in r(\boldsymbol{A})$, then $e\backslash z = z = z/e$.*

(iv) *If $\boldsymbol{A}$ is commutative, then $r(\boldsymbol{A}) = R(\boldsymbol{A})$.*

A *residuated po-semigroup* $\boldsymbol{A} = (A, \leq, \cdot, \backslash, /)$ is a po-semigroup $(A, \leq, \cdot)$ with two binary operations $\backslash, /$ which satisfy the residuation law

$$x \cdot y \leq z \text{ iff } x \leq z/y \text{ iff } y \leq x\backslash z \tag{1}$$

for all $x, y, z \in A$. Thus, in the po-semigroup $(A, \leq, \cdot)$, all elements are (weakly) residuable. A *residuated po-monoid* is a structure $\boldsymbol{A} = (A, \leq, \cdot, \backslash, /, e)$ such that $(A, \leq, \cdot, \backslash, /)$ is a residuated po-semigroup and $(A, \cdot, e)$ is a monoid.

A *quantum B-algebra* [13], [15] is a structure $\boldsymbol{A} = (A, \leq, \backslash, /)$ where $\leq$ is a partial order and $\backslash, /$ are binary operations such that for all $x, y, z \in A$:

$$x\backslash(z/y) = (x\backslash z)/y; \tag{2}$$

$$x \leq y \text{ implies } z\backslash x \leq z\backslash y \text{ and } x/z \leq y/z; \tag{3}$$

$$x \leq z/y \text{ iff } y \leq x\backslash z. \tag{4}$$

A *unital* quantum B-algebra $\boldsymbol{A} = (A, \leq, \backslash, /, u)$ is a quantum B-algebra $(A, \leq, \backslash, /)$ with constant u satisfying $u\backslash x = x = x/u$ for all $x \in A$. In this case, $x \leq y$ iff $u \leq x\backslash y$ iff $u \leq y/x$. If, moreover, the unit u is the greatest element of $(A, \leq)$, then $\boldsymbol{A}$ is a *pseudo-BCK-algebra* [7] (or a *biresiduation algebra* [17]).

We say that a quantum B-algebra $\boldsymbol{A} = (A, \leq, \backslash, /)$ is

(i) *commutative*[1] if $x\backslash y = y/x$ for all $x, y \in A$;

(ii) *two-sided* if $x \leq x/y$ and $x \leq y\backslash x$ for all $x, y \in A$;

(iii) *idempotent* if, for all $x, y \in A$, $x \leq y$ iff $x \leq y/x$ iff $x \leq x\backslash y$.

A routine application of (1) shows that if $(A, \leq, \cdot, \backslash, /)$ is a residuated po-semigroup, then $(A, \leq, \backslash, /)$ is a quantum B-algebra. More generally, we have:

Proposition 2. *Let $\boldsymbol{A} = (A, \leq, \cdot)$ be a po-semigroup. Let B be a non-empty subset of $R(\boldsymbol{A})$ such that $a, b \in B$ implies $a\backslash b, b/a \in B$. Then $\boldsymbol{B} = (B, \leq, \backslash, /)$ equipped with the induced order relation $\leq$ and binary operations $\backslash, /$ is a quantum B-algebra. Moreover, we have:*

(i) *if the po-semigroup $\boldsymbol{A}$ is commutative, two-sided or idempotent, then the quantum B-algebra $\boldsymbol{B}$ is commutative, two-sided or idempotent, respectively;*

(ii) *if $\boldsymbol{A}$ is a po-monoid with unit e and if $e \in B$, then $(B, \leq, \backslash, /, e)$ is a unital quantum B-algebra;*

(iii) *if $\boldsymbol{A}$ is an integral po-monoid with unit $e \in B$, then $(B, \leq, \backslash, /, e)$ is a pseudo-BCK-algebra.*

Proof. By Lemma 1 (i) and (ii) we obtain (2) and (3), while (4) follows directly from the definition of residuable elements. The remaining parts of the statement are evident. □

Let $\boldsymbol{A} = (A, \leq, \backslash, /)$ be a quantum B-algebra. For any non-empty word $\alpha = a_1 \ldots a_n$ of the free semigroup A^+ over A and any element $x \in A$ we write $\alpha\backslash x$ and x/α for

$$a_n\backslash(\ldots\backslash(a_1\backslash x)\ldots) \text{ and } (\ldots(x/a_n)/\ldots)/a_1,$$

[1] In BCK- and pseudo-BCK-algebras, the adjective "commutative" traditionally has another meaning. Namely, a pseudo-BCK-algebra is called commutative if it satisfies the identity $x/(y\backslash x) = (y/x)\backslash y$ that, however, does not entail $x\backslash y = y/x$.

respectively. When allowing the empty word ε, we have $\varepsilon\backslash x = x = x/\varepsilon$ for any $x \in A$. It is easily seen that in analogy with (2) and (3) we have

$$\alpha\backslash(x/\beta) = (\alpha\backslash x)/\beta, \tag{5}$$

$$x \le y \text{ implies } \alpha\backslash x \le \alpha\backslash y \text{ and } x/\alpha \le y/\alpha, \tag{6}$$

for all words α, β of the free monoid A^* over A and $x, y \in A$. Moreover, by repeatedly using (4) and (5) we get $a_1 \le x/a_2 \dots a_n$ iff $a_n \le a_1 \dots a_{n-1}\backslash x$, for any $\alpha = a_1 \dots a_n \in A^+$ and $x \in A$, which allows us to write

$$\alpha \le x, \tag{7}$$

just as in [14]. The following generalizations of the residuation law (1) and of transitivity then hold for all $\alpha, \beta \in A^+$ and $x, y \in A$:

$$\alpha\beta \le x \text{ iff } \alpha \le x/\beta \text{ iff } \beta \le \alpha\backslash x, \tag{8}$$

$$\alpha \le x \text{ and } x \le y \text{ imply } \alpha \le y. \tag{9}$$

The relation (7) induces, again as in [14], a Galois connection between $\mathcal{P}(A^+)$ and $\mathcal{P}(A)$. We write

$$L^\uparrow = \{x \in A \mid \alpha \le x \text{ for all } \alpha \in L\}$$

for any $L \subseteq A^+$. If L is $\{\alpha\}$ for some $\alpha \in A^+$, we write simply $\alpha^\uparrow$ instead of $\{\alpha\}^\uparrow$.

It is obvious by (9) that for any $L \subseteq A$, $L^\uparrow$ is an up-set in $(A, \le)$ and, for any $a \in A$, $a^\uparrow = \{x \in A \mid a \le x\}$ is the principal order-filter of $(A, \le)$ generated by a.

The converse of Proposition 2 is also true:

Theorem 3. *Let $\boldsymbol{A} = (A, \le, \backslash, /)$ be a quantum B-algebra. There exists a po-semigroup $\boldsymbol{\mathcal{M}} = (\mathcal{M}, \sqsubseteq, *)$ and a non-empty subset $\mathcal{A}$ of $R(\boldsymbol{\mathcal{M}})$ equipped with induced binary operations $\backslash\!\!\backslash$ and $/\!\!/$ such that $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \sqsubseteq, \backslash\!\!\backslash, /\!\!/)$ is a quantum B-algebra isomorphic to $\boldsymbol{A}$. In addition,*

(i) *the subset $\mathcal{A}$ is meet-dense in $\mathcal{M}$ and $\mathcal{M}$ is multiplicatively generated by $\mathcal{A}$;*
(ii) *$\boldsymbol{A}$ is a commutative quantum B-algebra if and only if $\boldsymbol{\mathcal{M}}$ is a commutative po-semigroup;*
(iii) *$\boldsymbol{A}$ is a two-sided quantum B-algebra if and only if $\boldsymbol{\mathcal{M}}$ is a two-sided po-semigroup;*
(iv) *if $\boldsymbol{A}$ is an idempotent quantum B-algebra, then $\mathcal{A} \subseteq Id(\boldsymbol{\mathcal{M}})$, where $Id(\boldsymbol{\mathcal{M}})$ is the set of idempotent elements of the po-semigroup $\boldsymbol{\mathcal{M}}$;*
(v) *$\boldsymbol{A}$ is an idempotent and commutative quantum B-algebra if and only if $\boldsymbol{\mathcal{M}}$ is an idempotent and commutative po-semigroup;*
(vi) *$\boldsymbol{A}$ is an idempotent, commutative and two-sided quantum B-algebra if and only if $\boldsymbol{\mathcal{M}}$ is an idempotent, commutative and two-sided po-semigroup, in which case $(\mathcal{M}, \sqsubseteq)$ is a lower semilattice with meet operation $*$.*

Proof. The proof consists of several steps.

(a) Following [2], we take $\mathcal{M} = \{\alpha^\uparrow \mid \alpha \in A^+\}$ and equip it with a binary operation $*$ as follows:

$$\alpha^\uparrow * \beta^\uparrow = (\alpha\beta)^\uparrow.$$

It is easy to verify that $*$ is well-defined and that $\boldsymbol{\mathcal{M}} = (\mathcal{M}, \supseteq, *)$ is a po-semigroup. (Clearly, $(\mathcal{M}, \subseteq, *)$ is a po-semigroup, too, but we will see below that it is better to take $\supseteq$ rather than $\subseteq$.)

First, we observe that

$$\begin{aligned}\alpha^\uparrow * \beta^\uparrow &= \{z \in A \mid xy \le z \text{ for some } x \in \alpha^\uparrow, y \in \beta^\uparrow\}\\ &= \bigcup\{(xy)^\uparrow \mid x \in \alpha^\uparrow, y \in \beta^\uparrow\}\end{aligned}$$

for all $\alpha, \beta \in A^+$. Indeed, for every $z \in A$, if $z \in (\alpha\beta)^\uparrow$, i.e. $\alpha\beta \le z$, then letting $x = z/\beta$ and $y = x\backslash z$, we have $\alpha \le z/\beta = x$ and $\beta \le x\backslash z = y$ by (8), thus $xy \le z$ where $x \in \alpha^\uparrow$ and $y \in \beta^\uparrow$. Conversely, if $xy \le z$ for some $x \in \alpha^\uparrow$ and $y \in \beta^\uparrow$, then $\alpha \le x$ and $x \le z/y$ imply $\alpha \le z/y$ by (9), thus $y \le \alpha\backslash z$, which along with $\beta \le y$ yields $\beta \le \alpha\backslash z$ again by (9), and so $\alpha\beta \le z$.

This shows that Fleischer's product [2] on $\mathcal{M}$ is the restriction of Rump's product [13], [15] on $\mathcal{U}$, the set of all up-sets in $(A, \le)$, which is defined by

$$\begin{aligned}X * Y &= \{z \in A \mid x\backslash z \in Y \text{ for some } x \in X\}\\ &= \{z \in A \mid z/y \in X \text{ for some } y \in Y\}\\ &= \{z \in A \mid xy \le z \text{ for some } x \in X, y \in Y\}\\ &= \bigcup\{(xy)^\uparrow \mid x \in X, y \in Y\},\end{aligned}$$

for $X, Y \in \mathcal{U}$.

Second, let us check that $*$ is associative and order-preserving. Let $\alpha, \beta, \gamma \in A^+$. Then

$$\begin{aligned}(\alpha^\uparrow * \beta^\uparrow) * \gamma^\uparrow &= (\alpha\beta)^\uparrow * \gamma^\uparrow = (\alpha\beta\gamma)^\uparrow\\ &= \alpha^\uparrow * (\beta\gamma)^\uparrow = \alpha^\uparrow * (\beta^\uparrow * \gamma^\uparrow)\end{aligned}$$

and from $\beta^\uparrow \supseteq \gamma^\uparrow$ we obtain

$$\begin{aligned}\alpha^\uparrow * \beta^\uparrow &= \bigcup\{(xy)^\uparrow \mid x \in \alpha^\uparrow, y \in \beta^\uparrow\}\\ &\supseteq \bigcup\{(xy)^\uparrow \mid x \in \alpha^\uparrow, y \in \gamma^\uparrow\}\\ &= \alpha^\uparrow * \gamma^\uparrow\end{aligned}$$

and similarly, $\beta^\uparrow * \alpha^\uparrow \supseteq \gamma^\uparrow * \alpha^\uparrow$.

(b) For all $\alpha, \gamma \in A^+$ and $b \in A$ we have

$$\gamma^\uparrow \supseteq (\alpha\backslash b)^\uparrow \text{ iff } \gamma \le \alpha\backslash b \text{ iff } \alpha\gamma \le b \text{ iff } (\alpha\gamma)^\uparrow \supseteq b^\uparrow,$$

and symmetrically,

$$\gamma^{\uparrow} \supseteq (b/\alpha)^{\uparrow} \text{ iff } \gamma \leq b/\alpha \text{ iff } \gamma\alpha \leq b \text{ iff } (\gamma\alpha)^{\uparrow} \supseteq b^{\uparrow}.$$

Thus, in the po-semigroup $\boldsymbol{\mathcal{M}} = (\mathcal{M}, \supseteq, *)$, the residuals $\alpha^{\uparrow} \backslash\!\backslash b^{\uparrow} = (\alpha\backslash b)^{\uparrow}$ and $b^{\uparrow} /\!/ \alpha^{\uparrow} = (b/\alpha)^{\uparrow}$ exist for all $\alpha \in A^{+}$ and $b \in A$, since $\alpha\backslash b$ and b/α are again elements of A.

In particular, $a^{\uparrow} \backslash\!\backslash b^{\uparrow} = (a\backslash b)^{\uparrow}$ and $b^{\uparrow} /\!/ a^{\uparrow} = (b/a)^{\uparrow}$ exist for all $a, b \in A$. Since $a \leq b$ iff $a^{\uparrow} \supseteq b^{\uparrow}$, it follows that $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \supseteq, \backslash\!\backslash, /\!/)$, where $\mathcal{A} = \{a^{\uparrow} \mid a \in A\}$, is a quantum B-algebra which is an isomorphic copy of $\boldsymbol{A} = (A, \leq, \backslash, /)$.

(c) The set $\mathcal{A}$ has, among others, the following notable properties:

- $\mathcal{A}$ is meet-dense in the poset $(\mathcal{M}, \supseteq)$ because for every $\alpha^{\uparrow} \in \mathcal{M}$ we have $\alpha^{\uparrow} = \bigcup\{x^{\uparrow} \mid x \in \alpha^{\uparrow}\}$, whence $\alpha^{\uparrow} = \inf_{(\mathcal{M},\supseteq)}\{x^{\uparrow} \mid x \in \alpha^{\uparrow}\}$;
- $\mathcal{A}$ generates the semigroup $(\mathcal{M}, *)$ because $\alpha^{\uparrow} = a_1^{\uparrow} * \ldots * a_n^{\uparrow}$ when $\alpha = a_1 \ldots a_n$;
- $\mathcal{A} \subseteq R(\boldsymbol{\mathcal{M}})$ because, as proved in (b), both $\alpha^{\uparrow} \backslash\!\backslash b^{\uparrow}$ and $b^{\uparrow} /\!/ \alpha^{\uparrow}$ exist and belong to $\mathcal{A}$, for all $\alpha^{\uparrow} \in \mathcal{M}$ and $b^{\uparrow} \in \mathcal{A}$.

(d) It remains to prove that $\boldsymbol{A}$, which is isomorphic to $\boldsymbol{\mathcal{A}}$, and $\boldsymbol{\mathcal{M}}$ satisfy (ii)–(vi). If the po-semigroup $\boldsymbol{\mathcal{M}}$ is commutative, two-sided or idempotent, then by Proposition 2 (i) we have that the quantum B-algebra $\boldsymbol{\mathcal{A}}$ and hence also $\boldsymbol{A}$ has the respective property, too. Hence the "if" parts of (ii), (iii), (v) and (vi) are evident.

(ii) Assume that the quantum B-algebra $\boldsymbol{A}$ is commutative. Then, by (8), for all $x, y, z \in A$ we have $xy \leq z$ iff $x \leq z/y = y\backslash z$ iff $yx \leq z$. Thus $(xy)^{\uparrow} = (yx)^{\uparrow}$, which yields

$$\begin{aligned}\alpha^{\uparrow} * \beta^{\uparrow} &= \bigcup\{(xy)^{\uparrow} \mid x \in \alpha^{\uparrow}, y \in \beta^{\uparrow}\} \\ &= \bigcup\{(yx)^{\uparrow} \mid x \in \alpha^{\uparrow}, y \in \beta^{\uparrow}\} \\ &= \beta^{\uparrow} * \alpha^{\uparrow}\end{aligned}$$

for all $\alpha, \beta \in A^{+}$, proving that $\boldsymbol{\mathcal{M}}$ is a commutative po-semigroup.

(iii) Assume that the quantum B-algebra $\boldsymbol{A}$ is two-sided. Then for all $x, y \in A$ we have $x \leq x/y$, which is equivalent to $xy \leq x$, whence $(xy)^{\uparrow} \supseteq x^{\uparrow}$. Then

$$\begin{aligned}\alpha^{\uparrow} * \beta^{\uparrow} &= \bigcup\{(xy)^{\uparrow} \mid x \in \alpha^{\uparrow}, y \in \beta^{\uparrow}\} \\ &\supseteq \bigcup\{x^{\uparrow} \mid x \in \alpha^{\uparrow}\} \\ &= \alpha^{\uparrow}\end{aligned}$$

for all $\alpha, \beta \in A^{+}$. Similarly, $\beta^{\uparrow} * \alpha^{\uparrow} \supseteq \alpha^{\uparrow}$. Thus the po-semigroup $\boldsymbol{\mathcal{M}}$ is two-sided.

(iv) Assume that the quantum B-algebra $\boldsymbol{A}$ is idempotent. Then $a^{\uparrow} * a^{\uparrow} = a^{\uparrow}$ for every $a \in A$ because $aa \leq x$ iff $a \leq a\backslash x$ iff $a \leq x$ for every $x \in A$ by the definition of idempotency. Thus $\mathcal{A}$ is a subset of $Id(\boldsymbol{\mathcal{M}})$, the set of idempotent elements of $\boldsymbol{\mathcal{M}}$.

(v) Assume that the quantum B-algebra $\boldsymbol{A}$ is idempotent and commutative. Then $\boldsymbol{\mathcal{M}}$ is a commutative po-semigroup by (ii) and $\mathcal{A} \subseteq Id(\boldsymbol{\mathcal{M}})$ by (iv). For every $\alpha = a_1 \ldots a_n \in A^{+}$ we have $\alpha^{\uparrow} * \alpha^{\uparrow} = a_1^{\uparrow} * \ldots * a_n^{\uparrow} * a_1^{\uparrow} * \ldots * a_n^{\uparrow} = a_1^{\uparrow} * a_1^{\uparrow} * \ldots * a_n^{\uparrow} * a_n^{\uparrow} = a_1^{\uparrow} * \ldots * a_n^{\uparrow} = \alpha^{\uparrow}$. Thus $\boldsymbol{\mathcal{M}}$ is idempotent.

(vi) Finally, assume that the quantum B-algebra $\boldsymbol{A}$ is idempotent, commutative and two-sided. By (iii) and (v) we know that $\boldsymbol{\mathcal{M}}$ is an idempotent, commutative and two-sided po-semigroup; in other words, $(\mathcal{M}, *)$ is a semilattice. To complete the proof we need to show that $\alpha^{\uparrow} \supseteq \beta^{\uparrow}$ iff $\alpha^{\uparrow} = \alpha^{\uparrow} * \beta^{\uparrow}$, for all $\alpha, \beta \in A^{+}$. But this easily follows from $\boldsymbol{\mathcal{M}}$ being two-sided: If $\alpha^{\uparrow} \supseteq \beta^{\uparrow}$, then $\alpha^{\uparrow} = \alpha^{\uparrow} * \alpha^{\uparrow} \supseteq \alpha^{\uparrow} * \beta^{\uparrow} \supseteq \alpha^{\uparrow}$, so $\alpha^{\uparrow} = \alpha^{\uparrow} * \beta^{\uparrow}$. Conversely, if $\alpha^{\uparrow} = \alpha^{\uparrow} * \beta^{\uparrow}$, then clearly $\alpha^{\uparrow} \supseteq \beta^{\uparrow}$. □

Concerning (iv), there exist idempotent quantum B-algebras or even pseudo-BCK-algebras $\boldsymbol{A}$ such that the po-monoid $\boldsymbol{\mathcal{M}}$ is not idempotent. (A concrete example can be the algebra $\boldsymbol{D}_4$ in [17], p. 435.)

Concerning the difference between (v) and (vi), in both cases $(\mathcal{M}, *)$ is a semilattice, so the rule $\alpha^{\uparrow} \preceq \beta^{\uparrow}$ iff $\alpha^{\uparrow} = \alpha^{\uparrow} * \beta^{\uparrow}$ defines a partial order on $\mathcal{M}$, but if $\boldsymbol{A}$ and/or $\boldsymbol{\mathcal{M}}$ is not two-sided, then $\preceq$ differs from $\sqsubseteq$ (i.e., from $\supseteq$ in the proof above).

Let us determine what happens when we work with unital quantum B-algebras. So, let $\boldsymbol{A} = (A, \leq, \backslash, /, u)$ be a unital quantum B-algebra. For every $\alpha \in A^{+}$ and $x \in A$ we have $u \leq \alpha\backslash x$ iff $\alpha \leq x/u = x$, and likewise $u \leq x/\alpha$ iff $\alpha \leq u\backslash x = x$. Hence we may write $\varepsilon \leq x$ iff $u \leq x$. Then $\varepsilon^{\uparrow} = u^{\uparrow} \in \mathcal{A}$ and $\boldsymbol{\mathcal{M}} = (\mathcal{M}, \supseteq, *, u^{\uparrow})$ is a po-monoid.

Theorem 4. *Let $\boldsymbol{A} = (A, \leq, \backslash, /, u)$ be a unital quantum B-algebra. There exists a po-monoid $\boldsymbol{\mathcal{M}} = (\mathcal{M}, \sqsubseteq, *, e)$ and a non-empty subset $\mathcal{A}$ of $R(\boldsymbol{\mathcal{M}})$ equipped with induced binary operations $\backslash\!\backslash$ and $/\!/$ such that $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \sqsubseteq, \backslash\!\backslash, /\!/, e)$ is a unital quantum B-algebra isomorphic to $\boldsymbol{A}$. Mutatis mutandis, (i)–(vi) of Theorem 3 hold true for $\boldsymbol{A}$ and $\boldsymbol{\mathcal{M}}$. Moreover, if $\boldsymbol{A}$ is a pseudo-BCK-algebra, then $\boldsymbol{\mathcal{M}}$ is an integral po-monoid.*

Proof. With the exception of the final statement, this follows immediately from Proposition 2 and Theorem 3. The final statement is also quite obvious because when $\boldsymbol{A}$ is a pseudo-BCK-algebra, then u is the greatest element and $\varepsilon^{\uparrow} = u^{\uparrow} = \{u\}$ is the smallest up-set, so that $\boldsymbol{\mathcal{M}}$ is an integral po-monoid. $\square$

II. Quantum B-algebras and Fleischer po-semigroups

The preceding Theorem 3 suggests that quantum B-algebras can be identified with a special class of po-semigroups.

Therefore, in the light of Theorem 3, and since our construction is clearly inspired by that by Fleischer [2], we define a *Fleischer po-semigroup* to be a pair $(\boldsymbol{M}, A)$ where $\boldsymbol{M} = (M, \leq, \cdot)$ is a po-semigroup and A is a distinguished non-empty subset of $R(\boldsymbol{M})$ satisfying the following conditions:

- $a, b \in A$ implies $a\backslash b, b/a \in A$;
- A is meet-dense in the poset $(M, \leq)$;
- A generates the semigroup $(M, \cdot)$.

Let $(\boldsymbol{M}, A)$ be a Fleischer po-semigroup. From Proposition 2 we obtain that $\boldsymbol{A} = (A, \leq, \backslash, /)$ is a quantum B-algebra. Every $z \in M$ is a meet of elements of A, so $z = \inf\{a \in A \mid z \leq a\}$. Moreover, for every $\alpha = a_1 \dots a_n \in A^+$ and $x \in A$:

$$\alpha \leq x \text{ (in } \boldsymbol{A}) \text{ iff } a_1 \cdot \ldots \cdot a_n \leq x \text{ (in } \boldsymbol{M}). \tag{10}$$

Indeed, recalling Lemma 1 (ii), we have $a_1 \dots a_n \leq x$ in $\boldsymbol{A}$ iff $a_n \leq a_1 \dots a_{n-1}\backslash x = a_{n-1}\backslash(\dots\backslash(a_1\backslash x)\dots) = (a_1 \cdot \ldots \cdot a_{n-1})\backslash x$ iff $a_1 \cdot \ldots \cdot a_{n-1} \cdot a_n \leq x$ in $\boldsymbol{M}$.

Further, by (the proof of) Theorem 3 the quantum B-algebra $\boldsymbol{A}$ determines the Fleischer po-semigroup $(\boldsymbol{\mathcal{M}}, \mathcal{A})$, where $\boldsymbol{\mathcal{M}} = (\mathcal{M}, \supseteq, *)$. We aim at showing that the Fleischer po-semigroups $(\boldsymbol{M}, A)$ and $(\boldsymbol{\mathcal{M}}, \mathcal{A})$ are isomorphic (in the obvious sense).

In view of (10) we have

$$\alpha^{\uparrow} = \{x \in A \mid a_1 \cdot \ldots \cdot a_n \leq x\}$$

for every $\alpha = a_1 \dots a_n \in A^+$, whence we may define a map $\varphi \colon M \to \mathcal{M}$ simply by

$$\varphi(z) = \{x \in A \mid z \leq x\}.$$

Indeed, every $z \in M$ is of the form $z = a_1 \cdot \ldots \cdot a_n$ for some $a_1, \dots, a_n \in A$, in which case $z \leq x$ iff $a_1 \dots a_n \leq x$, and so $\varphi(z) = (a_1 \dots a_n)^{\uparrow}$. It is evident that $\varphi(z)$ is independent of the choice of $a_1, \dots, a_n$. Also note that $z = \inf \varphi(z)$. Then $u \leq v$ iff $\varphi(u) \supseteq \varphi(v)$ for all $u, v \in M$. Moreover, for any $\alpha = a_1 \dots a_n \in A^+$ we have $\alpha^{\uparrow} = \varphi(a_1 \cdot \ldots \cdot a_n)$, so the map φ is surjective. If $u, v \in M$, with $u = a_1 \cdot \ldots \cdot a_m$ and $v = b_1 \cdot \ldots \cdot b_n$, then

$$\begin{aligned} \varphi(u \cdot v) &= \{x \in A \mid u \cdot v \leq x\} \\ &= (a_1 \dots a_m b_1 \dots b_n)^{\uparrow} \\ &= (a_1 \dots a_m)^{\uparrow} * (b_1 \dots b_n)^{\uparrow} \\ &= \varphi(u) * \varphi(v). \end{aligned}$$

Thus φ is an isomorphism between the po-semigroups $\boldsymbol{M} = (M, \leq, \cdot)$ and $\boldsymbol{\mathcal{M}} = (\mathcal{M}, \supseteq, *)$. Finally, it is obvious that $\varphi(A) = \{a^{\uparrow} \mid a \in A\} = \mathcal{A}$, whence φ is the desired isomorphism between the Fleischer po-semigroups $(\boldsymbol{M}, A)$ and $(\boldsymbol{\mathcal{M}}, \mathcal{A})$.

We have seen before (Theorem 3) that if $\boldsymbol{A}$ is a quantum B-algebra with associated Fleischer po-semigroup $(\boldsymbol{\mathcal{M}}, \mathcal{A})$, then the quantum B-algebras $\boldsymbol{A}$ and $\boldsymbol{\mathcal{A}}$ are isomorphic. Therefore:

Theorem 5. *There is a one-to-one correspondence between (commutative, two-sided) quantum B-algebras and (commutative, two-sided) Fleischer po-semigroups.*

Lastly, we include unital quantum B-algebras and po-monoids. By a *Fleischer po-monoid* we mean a pair $(\boldsymbol{M}, A)$ where $\boldsymbol{M} = (M, \leq, \cdot, e)$ is a po-monoid and $((M, \leq, \cdot), A)$ is a Fleischer po-semigroup such that $e \in A$.

From Theorem 5 we obtain the following:

Theorem 6. *There is a one-to-one correspondence between (commutative) unital quantum B-algebras and (commutative) Fleischer po-monoids. There is a one-to-one correspondence between pseudo-BCK-algebras and integral Fleischer po-monoids. There is a one-to-one correspondence between BCK-algebras and integral commutative Fleischer po-monoids.*

III. Conclusion

Rump and Yang [13], [15] proved that quantum B-algebras correspond one-to-one to the so-called logical quantales. Our construction, mimicking Fleischer's construction for BCK-algebras [2], is more economical: Given a quantum B-algebra $\boldsymbol{A} = (A, \leq, \backslash, /)$, in order to build a po-semigroup whose set of residuable elements contains an isomorphic copy of the algebra $\boldsymbol{A}$, we consider only the up-sets generated by words $\alpha \in A^+$ (whereas all up-sets are considered in [13], [15]). Specifically, we take $\mathcal{M} = \{\alpha^{\uparrow} \mid \alpha \in A^+\}$ and prove that $\boldsymbol{\mathcal{M}} = (\mathcal{M}, \supseteq, *)$ is a po-semigroup, $\mathcal{A} = \{a^{\uparrow} \mid a \in A\} \subseteq R(\boldsymbol{\mathcal{M}})$ and $\boldsymbol{\mathcal{A}} = (\mathcal{A}, \supseteq, \mathbin{\backslash\!\!\backslash}, \mathbin{/\!\!/})$ is a quantum B-algebra isomorphic to $\boldsymbol{A}$. Thus $\boldsymbol{A}$ is fully determined by $(\boldsymbol{\mathcal{M}}, \mathcal{A})$.

Every (unital) quantum B-algebra can be embedded into a (unital) quantale [13], and in particular, every pseudo-BCK-algebra can be embedded into an integral quantale [10], [17]. One of possible proofs goes along the following lines: Let $\boldsymbol{A}$ be a quantum B-algebra. First, we construct the free quantale $\mathcal{P}(\boldsymbol{\mathcal{M}})$ over the semigroup $(\mathcal{M}, *)$. Then we choose a suitable nucleus j on $\mathcal{P}(\boldsymbol{\mathcal{M}})$ (see [12]) and construct the j-retraction $\mathcal{P}(\boldsymbol{\mathcal{M}})_j$; we may take $j(\mathcal{B}) = \{\alpha^\uparrow \in \mathcal{M} \mid \alpha^\uparrow \supseteq \beta^\uparrow \text{ for some } \beta^\uparrow \in \mathcal{B}\}$. Finally, the map $a \in A \mapsto j(\{a^\uparrow\}) \in \mathcal{P}(\mathcal{M})_j$ is the embedding in question.

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